Note:-

① If R_1 & R_2 are two equivalence relations on the set A, then $R_1 \cap R_2$ is also an equivalence relation on A.

i.e Intersection of two equivalence Relations is an equivalence relation. (Prove yourself)

② Union of two equivalence relations on the set A need not be an equivalence relation on A.

Give an example to show this fact:-

Take $A = \{1,2,3\}$ and consider

$R_1 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

$R_2 = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$

we observe that R_1 and R_2 are equivalence relation

Consider

$R_1 \cup R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}$

① This relation is Reflexive

② It is symmetric.

③ For transitive;

Consider $(3,1),(1,2) \in R_1 \cup R_2$

But $(3,2) \notin R_1 \cup R_2$

So It is not transitive.

So union of two equivalence Relations is not an equivalence relation.

classmate

Date

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(3) Let S be the set of all points in a plane and let R be a relation in S defined by

$$R = \{ (A, B) : d(A, B) < 2 \text{ units} \}$$

where $d(A, B)$ is the distance between the points A and B.

Show that R is reflexive and symmetric but not transitive.

functions

Recall:-

Definition:- let A and B be two non empty sets. Then any Rule f which corresponds to each $x \in A$, a unique element $f(x) \in B$,

means

for every input, there is a unique output.

i.e. If

$f: A \to B$

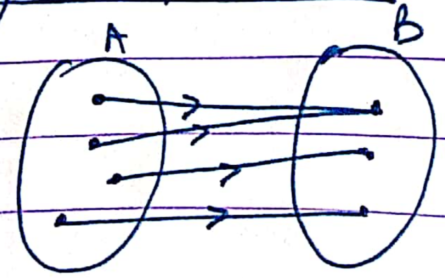
$A \to$ Domain of f

$B \to$ codomain of f.

and Range of $f = \{f(x) : x \in A\}$

Types of functions;

① Many one function:-



$f: A \to B$

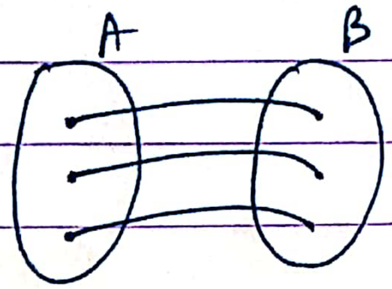
$f(x) = x^2$

$A = \{-1, 1, 2, 3\}$

$B = \{1, 4, 9\}$

Two Different elements have same image.

② ## One one function (Injective fn)



Different elements have different images.

To prove mathematically; we say function is one one if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

For example

Consider $f : N \longrightarrow N$ defined by

$$f(x) = 2x+3$$

To prove one one; we consider

$$f(x_1) = f(x_2)$$

$$\Rightarrow 2x_1 + 3 = 2x_2 + 3$$

$$\Rightarrow 2x_1 = 2x_2$$

$$\Rightarrow x_1 = x_2$$

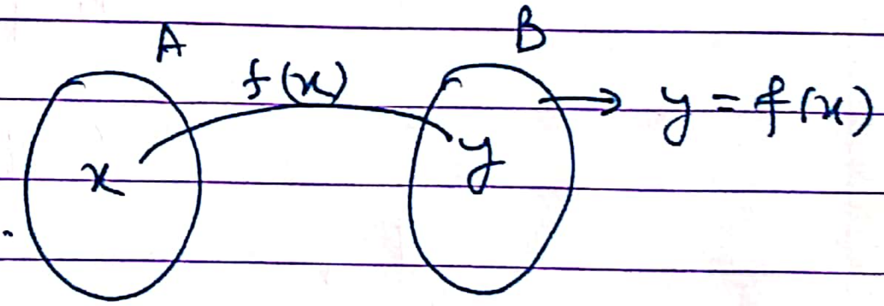
So $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\Rightarrow$ f is one one or injective function.

(3) **onto function or Surjective function**:

$f: A \longrightarrow B$ is said to be onto fn if for every element in codomain, there exists at least one element in domain such that $y = f(x)$.

i.e. for every $y \in B$, $\exists\ x \in A$ such that $y = f(x)$.



In other words, the codomain and range of function becomes same.

So fn f is onto if codomain = range.

Whenever we talk about onto function, we have to be very careful regarding the domain and range and codomain of f.

Consider the examples

$f_1 : N \longrightarrow N$

$f_1(x) = 2x$

$f_2 : N \longrightarrow E$ (set of even nos.)

$f_2(x) = 2x$

Find whether f_1 and f_2 are onto functions or not.

To prove f_1 to be onto, we should have

$$y = f_1(x)$$
$$y = 2x$$
$$x = \frac{y}{2}$$

Consider $y \in N$ (codomain) means y is any natural no.

Consider $y = 1, 2, 3, 4, 5, \ldots$

$\Rightarrow \quad x = \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}, \ldots$

$\Rightarrow x = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$

we observe that

x is taking fractions whereas x should be a natural numbers. as the $f_1 : N \longrightarrow N$. so $x \notin N$, for $y = 1, 3, 5, \ldots$

$\Rightarrow f_1$ is not onto..

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Consider $f_2 : N \longrightarrow E$

$f_2(x) : 2x$

For proving onto;

$$y = 2x$$

$$\Rightarrow x = \frac{y}{2}$$

Consider $y \in E \Rightarrow y$ is an even no.

$\Rightarrow y = 2, 4, 6, 8, 10, \text{----} \in E$

$\Rightarrow x = 1, 2, 3, 4, 10, \text{-----} \in N$

So for every value of $y \in E$, $\exists$ (there exist) $x \in N$ such that $y = f(x)$

$\Rightarrow f_2$ is onto.

So we have to be very careful about how the function are defined.

ONTO Function $\longrightarrow$ Surjective function.

4 Bijective function

A function which is one one and onto is known as a bijective function.

Bijective function have a property that they are invertible.

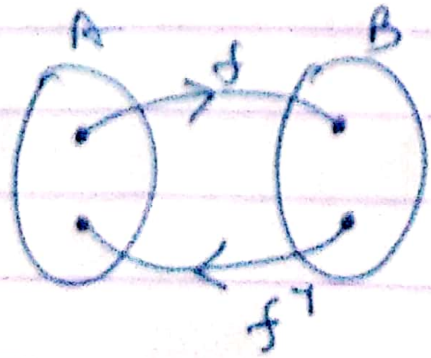
So if a function has to be invertible then it ~~has~~ has to be one one & onto otherwise we cannot talk about the inverse of the function.

So if

$f: A \longrightarrow B$ is a bijective function.

then $f^{-1}: B \longrightarrow A$

Roles of A & B are changed



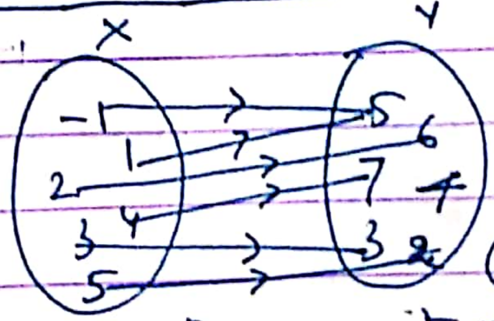
(bijective)

{one one correspondence}

Observation:-

$\sin x$, $\cos x$, $\tan x$ are many one and onto functions.
we can talk about their inverses only if we make them one one onto in the restricted intervals.

(5) Many one Into functions:-



(1) Many one fn.

(2) Into fn as there exit an element 4 in Y which is not the image of any element of X.

So fn $f: A \rightarrow B$ is said to be an into function if there exists even a single element in B which is having no preimage in A.

Take example

$A = \{2, 3, 5, 7\}$

$B = \{0, 1, 3, 5, 7\}$

Let

$f: A \rightarrow B$ $f(x) = x - 2$. Prove that it is an into function.

⑥ even function & odd function:

A fn is said to be even if $f(-x)=f(x)$

eg $f(x)=x^2$

$f(-x)=(-x)^2=x^2$

$\Rightarrow f(x)=f(-x)$

$f(x)=\sin^2 x=(\sin x)^2$

$f(-x)=\sin^2(-x)$

$=\{\sin(-x)\}^2$

$=(\sin x)^2$

odd fn:- If $f(-x)=-f(x)$

eg

$f(x)=x^3$

$f(-x)=(-x)^3=-x^3$

$=-f(x)$

$\Rightarrow f(-x)=-f(x) \Rightarrow f$ is an odd fn.

$f(x)=\tan^3 x$

$f(-x)=\tan^3(-x)=-\tan^3 x$

$=-f(x)$

$\Rightarrow f$ is an odd fn.

7 Equal functions:-

Two functions f and g are called equal iff

(a) domain of f = domain of g
(b) $f(x) = g(x) \ \forall \ x \in D_f$ or D_g.

For example let
$A = \{0, 2\}$
$B = \{1, 2, 3, 4, 5\}$
$f : A \longrightarrow B$ defined by $f(x) = 2x + 1$

$g : A \longrightarrow B$ defined by $g(x) = x^2 + 1$
$\forall \ x \in A$

Consider

$f(0) = 1$ $\quad$ $g(0) = 1$
$f(2) = 5$ $\quad$ $g(2) = 5$

so $f(x) = g(x) \ \forall \ x \in A \ (D_f \text{ or } D_g)$

F(1)

Questions to be discussed for functions

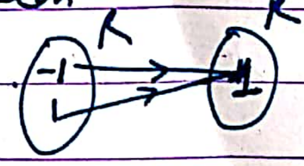
Consider the function $f: R \to R$ such that

(1) $f(x) = x^2$

It is a many one function

as $f(-1) = 1$

$f(1) = 1$



$\Rightarrow$ not one one fn.

Also it is ~~also~~ not onto as the $-$ve numbers in codomain does not have any preimage. In other words, we can not find out any element in the domain, whose images are negative. because output is always positive.

So $f(x) = x^2$ is neither one one not onto.

$f: R \to R$

(2) $f(x) = |x|$ is neither one one nor onto.

(3) $f: R \to R$; $f(x) = [x]$ is neither one one, nor onto.

(4) Consider signum fn. $f: R \to R$, defined by $f: R \to R$ such that

$$f(x) = \begin{cases} 1; & x > 0 \\ 0; & x = 0 \\ -1; & x < 0 \end{cases}$$

we observe that the function is many one function

Also Range of $f = \{-1, 0, 1\}$

$\Rightarrow$ Range $\subset$ Codomain

$\Rightarrow$ Function is not onto.

(5)

Show that the function $f: R \to R$, defined by $f(x) = 3 - 4x$ is one one onto and hence bijective. Also find f^{-1}.

$\rightarrow$ Solution:-

For one one we show $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

$\Rightarrow 3 - 4x_1 = 3 - 4x_2$

$4x_1 = 4x_2$

$\Rightarrow x_1 = x_2 \Rightarrow f$ is one one.

onto $f(x)$ will be an onto function if for every value of $y \in$ codomain, there must exist at least one $x \in$ domain, such that

$y = f(x)$

$y = 3 - 4x$

$4x = 3 - y$

$x = \frac{3-y}{4}$

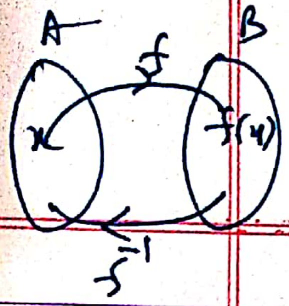
Consider $y \in$ Codomain R

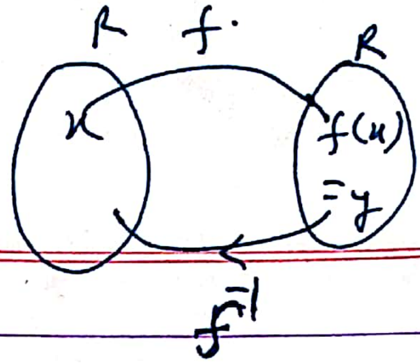
$\Rightarrow 3 - y \in R$

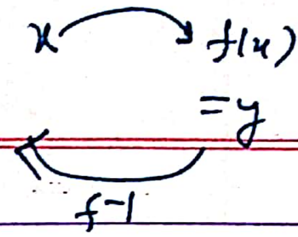
$\Rightarrow \frac{3-y}{4} \in R$

$\Rightarrow x \in R$ domain

$\Rightarrow f$ is onto hence Bijective







$f: A \to B$

$f^{-1}: B \to A$

$$f^{-1}(y) = x$$

So whatever x we have found out, that became f^{-1}.

$$f^{-1}: R \longrightarrow R$$

$$f^{-1}(y) = \frac{3-y}{4} \quad \text{or} \quad f^{-1}(x) = \frac{3-x}{4}$$

$$\text{or} \quad f^{-1}(z) = \frac{3-z}{4}$$

(6) Let $f: N \longrightarrow N$ defined by

$$f(x) = \begin{cases} x+1 & ; \text{ if } x \text{ is odd} \\ x-1 & ; \text{ if } x \text{ is even} \end{cases}$$

is bijective and find f^{-1}.

Solution:- For proving one one and onto function. i.e Bijective fn.

(1) one-one

Case I x_1, x_2 odd	Case 2 : x_1, x_2 even
Consider $f(x_1) = f(x_2)$	$f(x_1) = f(x_2)$
$x_1 + 1 = x_2 + 1$	$x_1 - 1 = x_2 - 1$
$\Rightarrow x_1 = x_2$	$x_1 = x_2$
fn is one one	fn is one one.

Case 3 x_1 even, x_2 odd

$f(x_1) = f(x_2)$

$x_1 - 1 = x_2 + 1$

$\boxed{x_1 - x_2 = 2}$

Difference of odd no & even no can not be even

$\Rightarrow$ It is a contradiction to our supposition.

$\Rightarrow$ x_1 even, x_2 odd (not possible)

similarly
Case 4 :- x_2 even, x_1 odd (not possible)

So fn is one one.

For onto

Consider $f(1) = 2$

$f(2) = 1$

$f(3) = 4$

$f(4) = 3$

For every odd number, there is an even image and for every even number, there is an odd image.

It means that codomain = Range $\Rightarrow$ f is onto.

$\Rightarrow$ function is one one and onto.

So f is Bijective and hence invertible.

For inverse consider

$y = f(x)$

$\Rightarrow y = x+1$ or $y = x-1$

x odd $\quad$ x, even

$\Rightarrow x = y-1$ $\quad$ $x = y+1$

$\Rightarrow y-1$ is odd $\quad$ $y+1$ is even

$\Rightarrow y$ is even $\quad$ y is odd

$$= f^{-1}(y) = \begin{cases} y-1, & y \text{ even} \\ y+1, & y \text{ odd} \end{cases}$$

(7) let $f: N \longrightarrow Y : f(x) = 4x^2+12x+15$

and Y = Range f.

Show that f is invertible and find f^{-1}!

$\rightarrow$ (1) To show one one $\quad$ (2) To show onto

(1) <u>one one</u>

$$f(x_1) = f(x_2)$$

$$4x_1^2+12x_1+15 = 4x_2^2+12x_2+15$$

$$4x_1^2+12x_1 = 4x_2^2+12x_2$$

$$4(x_1^2-x_2^2)+12(x_1-x_2) = 0$$

$$4(x_1-x_2)\{x_1+x_2+3\} = 0$$

Since $x_1 + x_2 + 3 \neq 0$ (F6)

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$ Hence f is one one

Also Range $f = Y \Rightarrow f$ is onto.

So f is invertible.

So $y = f(x)$

$\Rightarrow y = 4x^2 + 12x + 15$

$y = (2x)^2 + 2 \cdot 2x \cdot 3 + 9 - 9 + 15$

$y = (2x+3)^2 + 6$

$y - 6 = (2x+3)^2$

$\sqrt{y-6} = (2x+3)$

$\dfrac{\sqrt{y-6} - 3}{2} = x \Rightarrow \dfrac{\sqrt{y-6} - 3}{2} = f^{-1}(y)$

$\Rightarrow f^{-1} : Y \to N$ such that

$$\boxed{f^{-1}(y) = \frac{\sqrt{y-6} - 3}{2}}$$

OR

$f^{-1} : Y \to N$ such that

$$\boxed{f^{-1}(x) = \frac{\sqrt{x-6} - 3}{2}}$$

Q Let $A = \{1, 2, 3\}$ and let $f: A \to A$ defined
$f = \{(1,2), (2,3), (3,1)\}$
Find f^{-1}; if it exists.

Solution:-

$f(1) = 2$
$f(2) = 3$
$f(3) = 1$

we observe that different elements have different images. So f is one-one.

Also Range of $f = \{1,2,3\}$
$= A$

$\Rightarrow$ codomain = Range

$\Rightarrow$ f is onto

$\therefore$ f is one one and onto, Therefore it is invertible

$\Rightarrow$

$f^{-1}(2) = 1$
$f^{-1}(3) = 2$ So $f^{-1} = \{(2,1), (3,2), (1,3)\}$
$f^{-1}(1) = 3$

Observe f & f^{-1}

$f = \{(1,2), (2,3), (3,1)\}$

$f^{-1} = \{(2,1), (3,2), (1,3)\}$