SUBJECT – MATHEMATICS

CLASS-XII

CHAPTER :AREA UNDER CURVES

NCERT PDF OF CHAPTER -08 (APPLICATION OF INTEGRATION)

Chapter .7(AOI).pdf

INTRODUCTION

The problem of determining area of plane regions attracted the attention of Greek Geometers, especially Euclid(approx.300B.C.) and Archimedes(287-212B.C.).It is to compute the area of regular plane figures like triangle, square, trapezium, circles etc. Such formulae of elementary geometry allow us to calculate area of many simple figures.

However, these formulae are inadequate to find the area bounded/enclosed by curves. For that we need some more concepts of integral calculus

LEARNING OBJECTIVES…

- To evaluate the area between two functions using a difference of definite integrals
- To find the area bounded by a curve y = f(x) , x-axis and two ordinates x = a and x = b
- To find the area bounded by a curve x= f(y) , y-axis and two abscissa y = a and y = b
- To evaluate the area bounded by an irregular figure may be a curve, line, ordinates/abscissa and axes.
- In calculus, the integral of a function is an extension of the concept of a sum. the process of finding integrals is called integration. the process is usually used to find a measure of totality such as area, volume, mass, displacement, etc.
- The integral would be written $\int_a^b f(x)dx$. the $\int$ sign represents integration, a and b are the endpoints of the interval, $f(x)$ is the function we are integrating known as the integrand, and dx is a notation for the variable of integration. integrals discussed in this project are termed definite integrals.

Definite integral as the limit of a sum

Let f be a continuous function defined on close interval $[a, b]$.Assume that all the values taken by the function are non negative, so the graph the function is a curve above the x-axis. The definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x = a \ and \ x = b$.

Divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1]$, $[x_1, x_2]$,……………………$[x_{n-1}, x_n]$,where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$………… $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \frac{b-a}{h}$. We note that $n \to \infty, h \to 0$.

Definite integral as the limit of a sum……

The region PRSQP under consideration is the sum of n subregions, where each subregions is defined on subintervals $[x_{r-1}, x_r]$, r = 1,2,3,……..,n

From the side figure, we have,

Area of the rectangle(ABLC)

< area of the region (ABDCA)

<area of the rectangle(ABDM)…………….(1)

Evidently, x_r - $x_{r-1} \to 0. i.e, h \to 0$ all the three areas shown in (1) become nearly equal to each other.

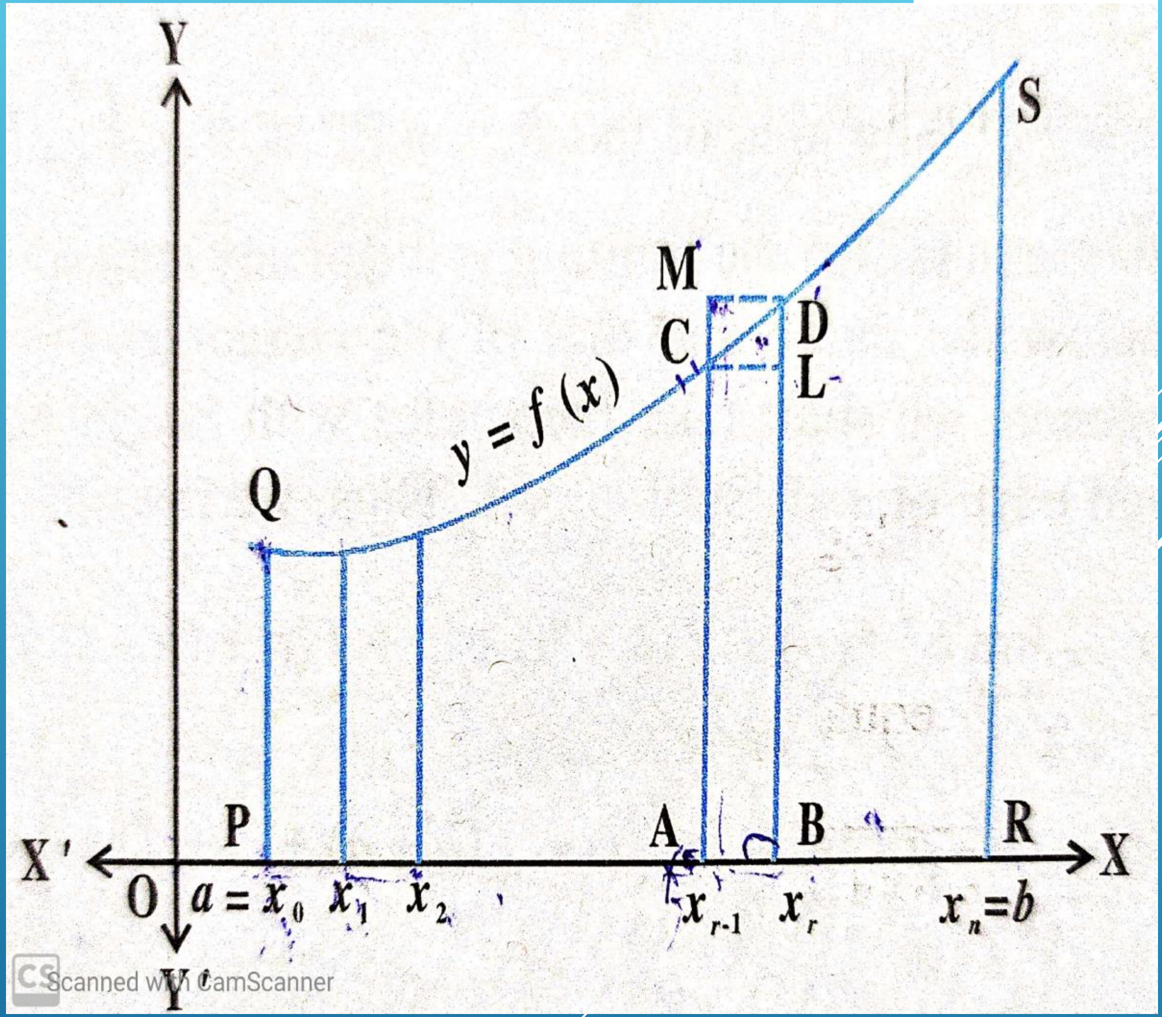
Now we form the following sums.

$s_n = h[f(x_0) + \cdots \ldots \ldots \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r)$…..(2)

and

$S_n = h[f(x_1) + f(x_2) \ldots \ldots \ldots \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r)$…(3)

here, S_n and s_n denote the sum of the areas of all upper rectangles and lower rectangles raised over subintervals $[x_{r-1}, x_r]$, r = 1,2,3,……..,n respectively.



Definite integral as the limit of a sum…….

In view of the inequality(1) for an arbitrary subintervals $[x_{r-1}, x_r]$, we have

$s_n <$ area of the region PRSQP$< S_n$ ……………(4)

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically we write

$\lim_{n\to\infty} S_n$ = $\lim_{n\to\infty} s_n$= area of the region PRSQP = $\int_a^b f(x)dx$ …………..(5)

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience , we shall take rectangles with height equal to that of the courve at the left hand edge of each subinterval. Thus , we rewrite (5) as

$\int_a^b f(x)dx$ = $\lim_{h\to 0} h[f(a) + f(a+h) + \ldots\ldots\ldots + f(a+(n-1)h]$

or $\int_a^b f(x)dx$ = $(\mathrm{b}-\mathrm{a}) \lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \ldots\ldots\ldots + f(a+(n-1)h]$ …… (6)

Where, $h = \frac{b-a}{n} \to 0 \; as \; n \to \infty$

The above expression (6) is known as the definition of definite integral as the limit of sum

Problem :(with marking scheme)

Evaluate

$$\int_{-2}^{2}(3x^2 - 2x + 4)dx$$

as the limit of sum

(CBSE Sample paper-2018)

Here, $f(x) = 3x^2 - 2x + 4$

$a = -2, b = 2 \qquad h = \frac{b-a}{n} \Rightarrow h = \frac{2+2}{n} \Rightarrow nh = 4$ **1**

We have $\int_a^b f(x)\,dx = \lim_{h \to 0} h\{f(a+h) + f(a+2h) + \ldots + f(a+nh)\}$

$\Rightarrow \qquad \int_{-2}^{2} f(x)\,dx = \lim_{h \to 0} h\{f(-2+h) + f(-2+2h) + f(-2+3h) + \ldots + f(-2+nh)\}$

Now, $f(-2+h) = 3(-2+h)^2 - 2(-2+h) + 4$

$= 3(4 - 4h + h^2) + 4 - 2h + 4 = 12 - 12h + 3h^2 + 4 - 2h + 4$

$= 3h^2 - 14h + 20$

$f(-2+2h) = 3(-2+2h)^2 - 2(-2+2h) + 4$

$= 3(4 - 8h + 4h^2) + 4 - 4h + 4$

$= 12 - 24h + 12h^2 + 4 - 4h + 4$

$= 12h^2 - 28h + 20$

Similarly, $f(-2+3h) = 27h^2 - 42h + 20$

| | |

| | |

| | |

$f(-2+nh) = 3(-2+nh)^2 - 2(-2+nh) + 4$

$= 12 - 12nh + 3n^2h^2 + 4 - 2nh + 4 = 3n^2h^2 - 14nh + 20$ **1**

Now, $\int_{-2}^{2}(3x^2 - 2x + 4)\,dx = \lim_{h \to 0} h\{(3h^2 - 14h + 20) + (12h^2 - 28h + 20) + (27h^2 - 42h + 20) + \ldots + (3n^2h^2 - 14nh + 20)\}$

$= \lim_{h \to 0} h\{3h^2(1^2 + 2^2 + 3^2 + \ldots + n^2) - 14h(1 + 2 + 3 + \ldots + n) + 20n\}$ **(2+½)**

$= \lim_{h \to 0}\left\{3h^3\frac{n(n+1)(2n+1)}{6} - 14h^2\frac{n(n+1)}{2} + 20nh\right\}$

$= \lim_{h \to 0}\left\{3\frac{nh(nh+h)(2nh+h)}{6} - 14\frac{nh(nh+h)}{2} + 20nh\right\}$

$= \lim_{h \to 0}\left\{\frac{4(4+h)(2\times4+h)}{2} - 7\times4(4+h) + 20\times4\right\}$

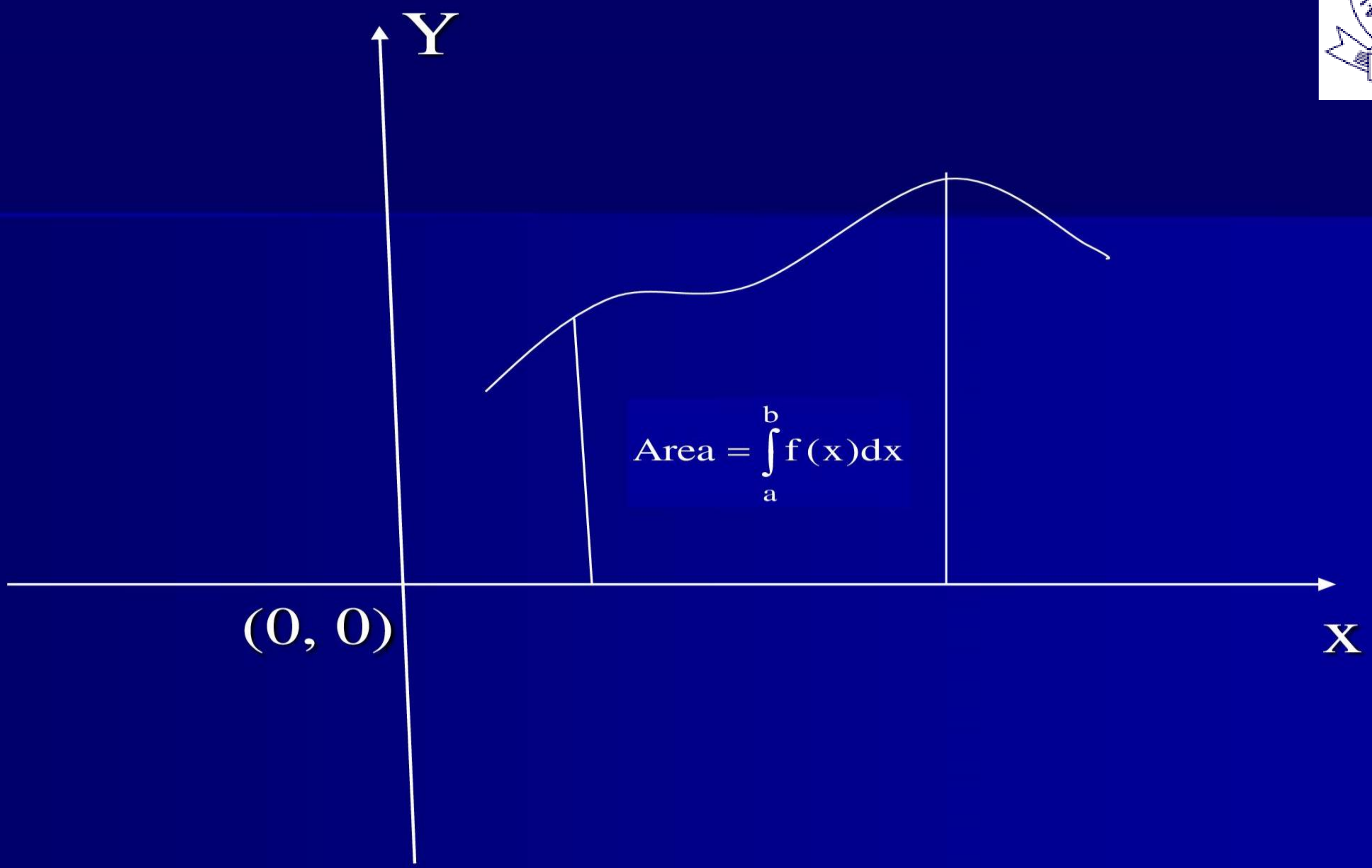
$= \{2\times4\times8 - 7\times4\times4 + 80\} = 64 - 112 + 80 = 32$ **1 + ½**

- The integration can be used to determine the area bounded by the plane curves, arc lengths volume and surface area of a region bounded by revolving a curve about a line.

AREA OF THE PLANE REGION

- We know that the area bounded by a Cartesian curve $y = f(x)$, x – axis, between lines $x = a$ & $x = b$ given by

$$\text{Area} = \int_a^b f(x)\,dx$$


Y
$\text{Area} = \int_a^b f(x)dx$
(0, 0)
X

AREA UNDER SIMPLE CURVES:

$\int_a^b f(x)dx$ *represents the area bounded by the curve* $y = f(x)$, $x -$ *axis ad two ordinates* $x = a$. *and* $x = b$

In the previous slide , we have studied definite integral as the limit of a sum and how to evaluate definite integral using Fundamental Theorem of Calculus.
Now we consider the easy and intuitive wa of finding the area bounded by the curve $y = f(x)$, $x - axis\ and\ the\ ordinates\ \boldsymbol{x} = \boldsymbol{a}\ and\ \boldsymbol{x} = b$. From the side figure we can think of area under the curve as composed of large number of very thin vertical stripes. Consider an arbitrary strip of height y and width dx, the dA(Area of the elementary strip) = $ydx, where, y = f(x)$.
This area is called the *elementary area* which is located at an arbitrary position within the region which is specified by some value of x between a and b .We can think of the total area A of the region between x-axis, ordinates $x = a, x = b$ and the curve $y = f(x)$ as the result of adding up the elementary areas of thin strips across the region PQRSP. Symbolically , we express

A = $\int_a^b dA = \int_a^b ydx = \int_a^b f(x)dx$

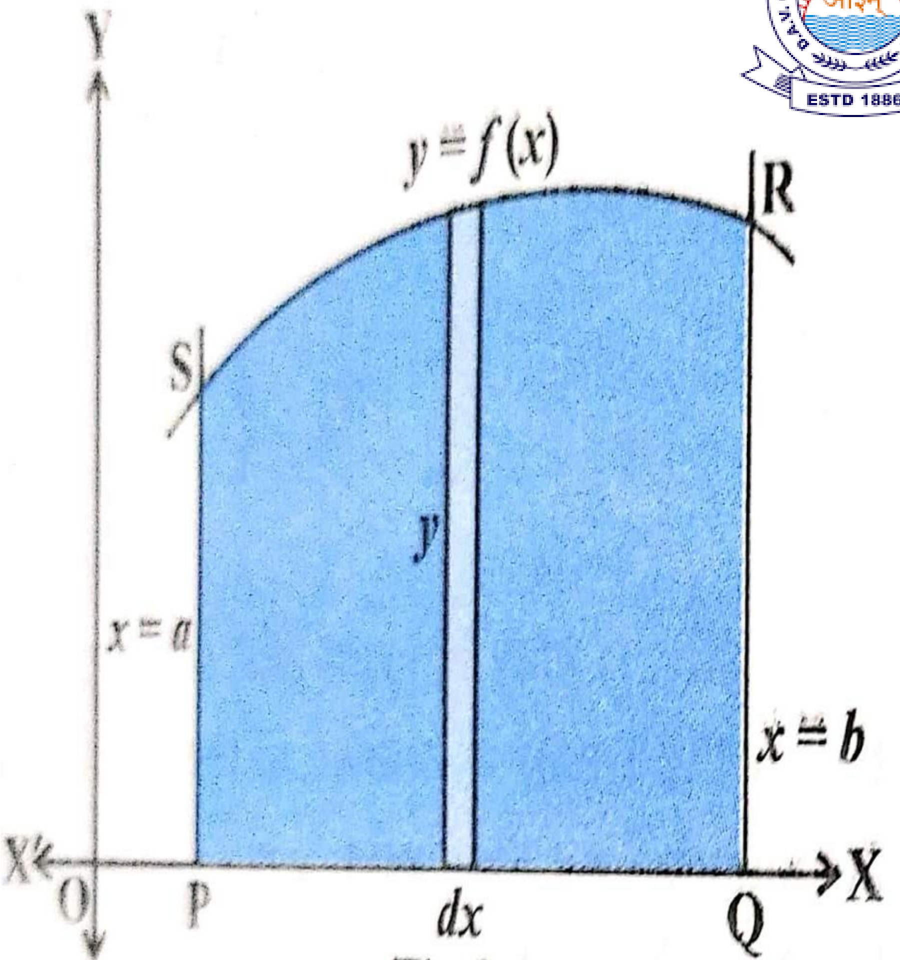


Fig 8.1

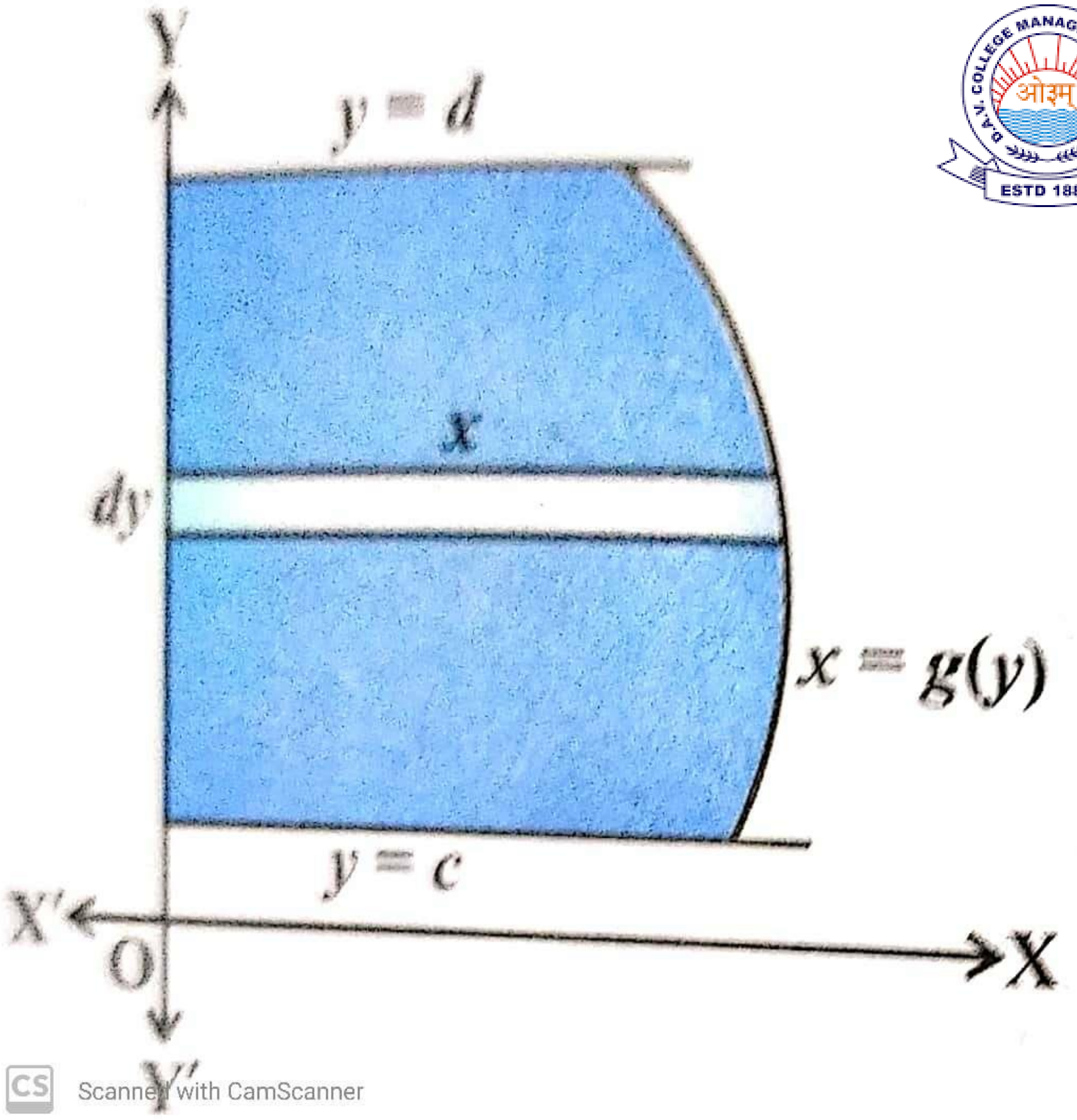
AREA UNDER SIMPLE CURVE:

$\int_a^b f(y)dy$ *represents the area bounded by the curve* $x = f(y), y - axis$ *ad two ordinates* $y = a$ *.and* $y = b$

The area A of the region bounded by the curve $x = g(y), y - axis$ and the lines $y = c, y = d$ is given by

A = $\int_c^d xdy = \int_c^d g(y)dy$

Here, we consider horizontal strips as shown in the side figure.



AREA UNDER SIMPLE CURVES:

If the position of the curve under consideration is below the x-axis

If the position of the curve under consideration is below the x-axis, then since $f(x) < 0$ from $x = a$ to $x = b$, as shown in the side figure, the area bounded by the curve , x-axis and the ordinates $x = a$, $x = b$ come out to be negative. But, it is only the numerical value of the area which is taken into consideration. Thus, if the area is negative, we take its absolute value, i.e., $\left|\int_a^b f(x)dx\right|$

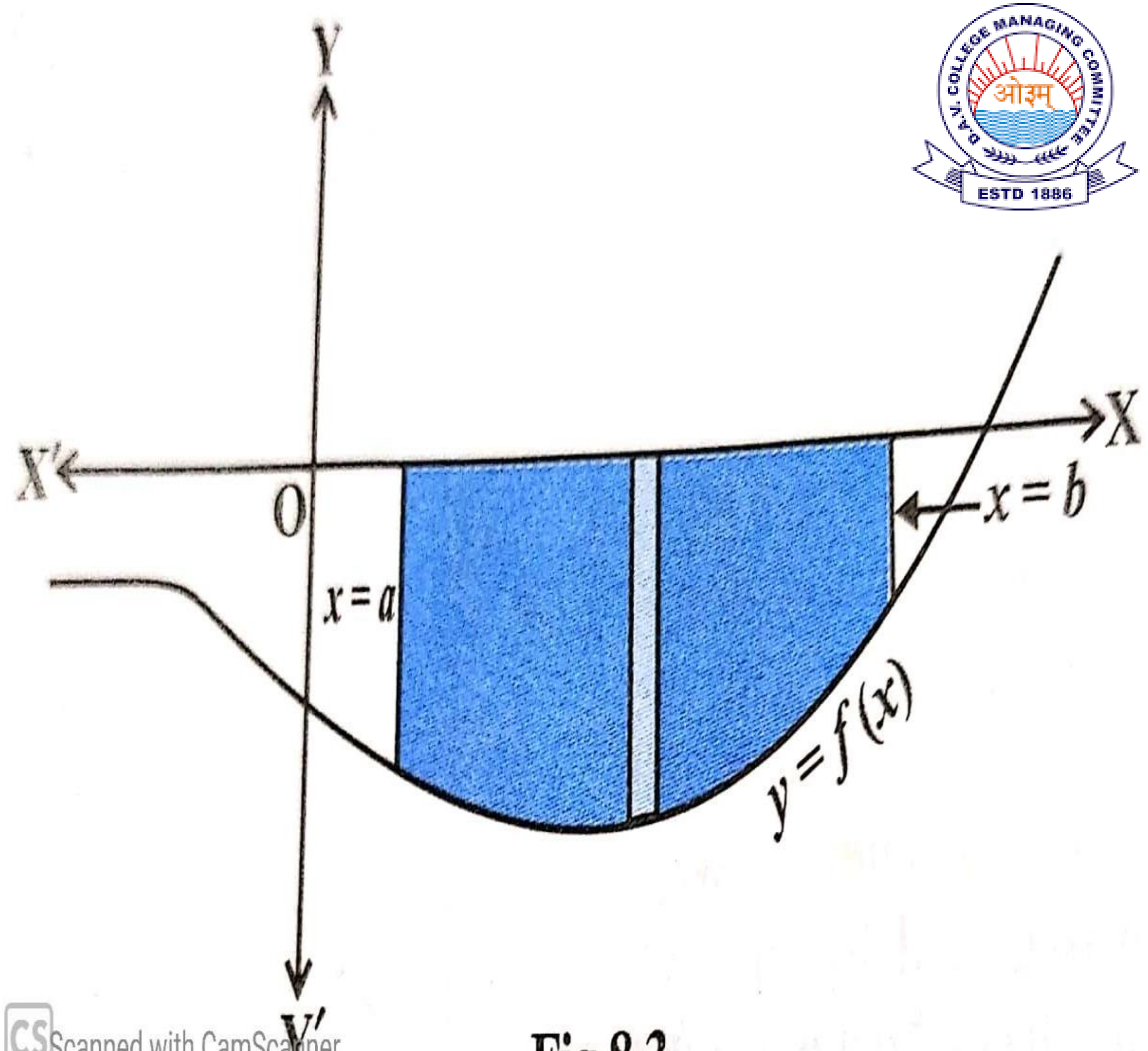


Fig 8.3

AREA UNDER SIMPLE CURVES:

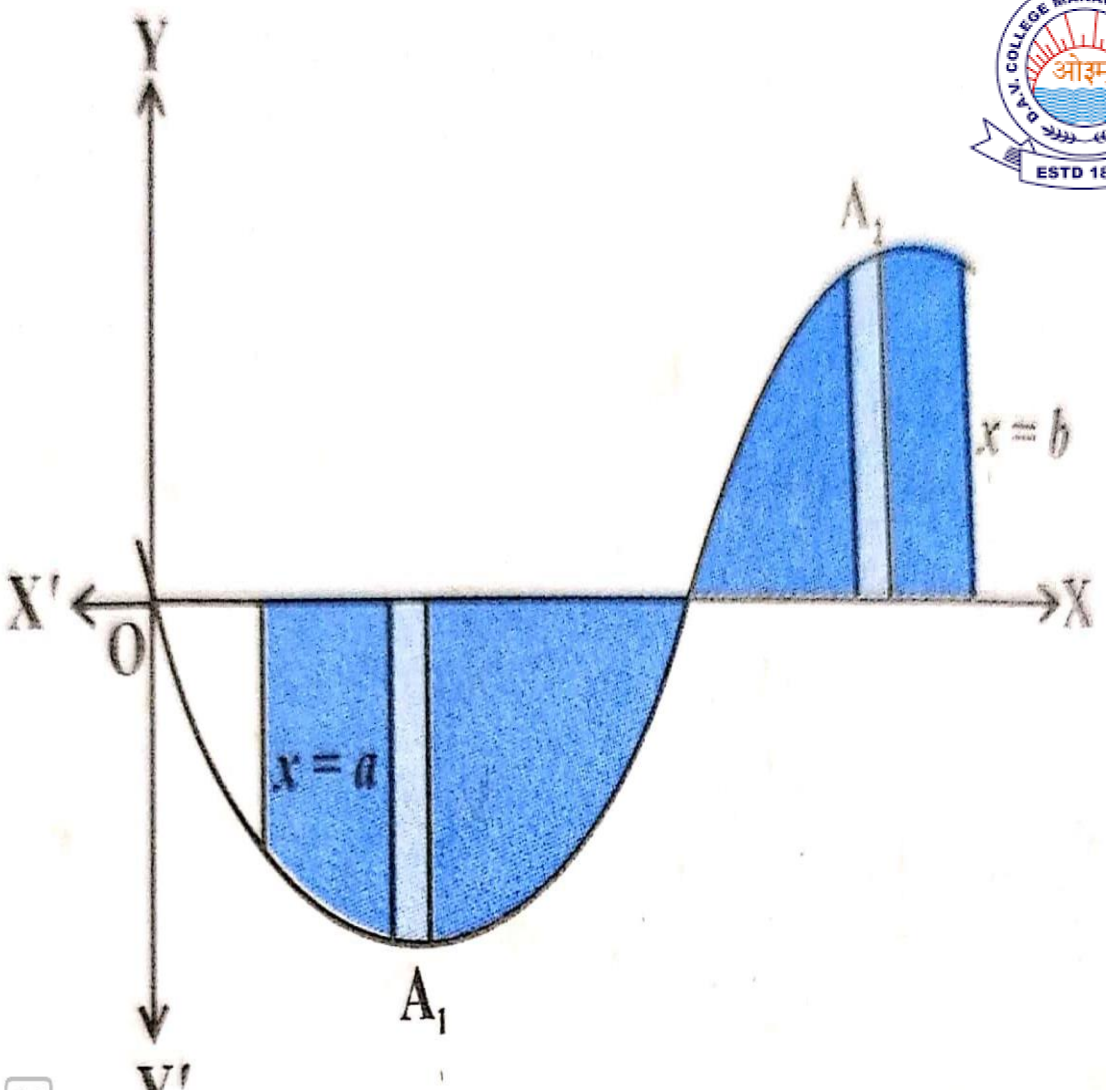
If some portion of the curve is above x-axis and some is below the x-axis

Generally, it may happen that some portion of the curve is above x-axis and some is below the x-axis as shown in the side figure.

Here, $A_1 < 0 \; and \; A_2 > 0$.

Therefore, the area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by

A = $|A_1|$ + A_2.



AREA UNDER SIMPLE CURVES:

The area bounded by two curves $y = f_1(x) and\ y = f_2(x)$ and two ordinates x= a and x=b

The area bounded by two curves $y = f_1(x) and\ y = f_2(x)$ and two ordinates x= a and x=b is given by

$$\int_a^b f_1(x)dx - \int_a^b f_2(x)dx$$

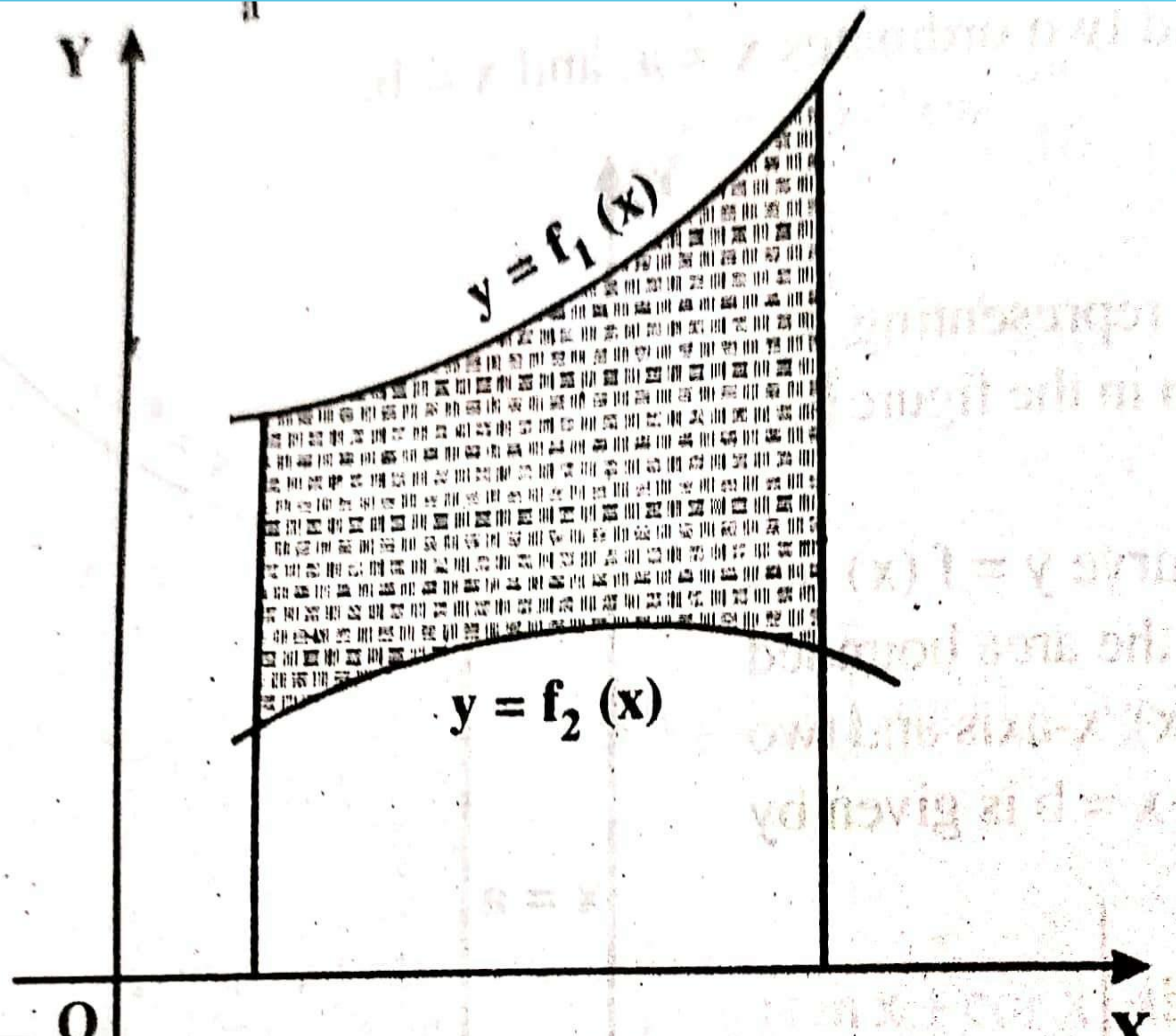


Fig. 3.6

AREA BETWEEN TWO CURVES

Suppose we are given two curves represented by $y = f(x), y = g(x)$, $where\ f(x) \geq \boldsymbol{g(x)}$ in $[\,a,\ b\,]$Here the point of intersection of these two curves are given $by\ x = a\ and\ x = b$ obtained by taking common values of y from the given equation of two curves.

For setting up a formula for the integral, it is convenient to take elementary area in the form of vertical strips. As indicated in the side figure, elementary strip has height $f(x) - g(x)$ and width dx so that the elementary area dA = $[f(x) - g(x)]dx$, and the total area A can be taken as

A = $\boldsymbol{\int_a^b [f(x) - g(x)]dx}$

Alternatively,

A= $[\ area\ bounded\ by\ y = f(x)\ , x - axis\ and\ the\ lines\ \boldsymbol{x} = a, x = \boldsymbol{b}] -$ $[\ area\ bounded\ by\ y = g(x), x - axis\ and\ the\ lines\ x = a, x = b]$

= $\boldsymbol{\int_a^b f(x)dx}$ - $\boldsymbol{\int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx},$

Where, $f(x) \geq \boldsymbol{g(x)}$ in $[\,a,\ b\,]$

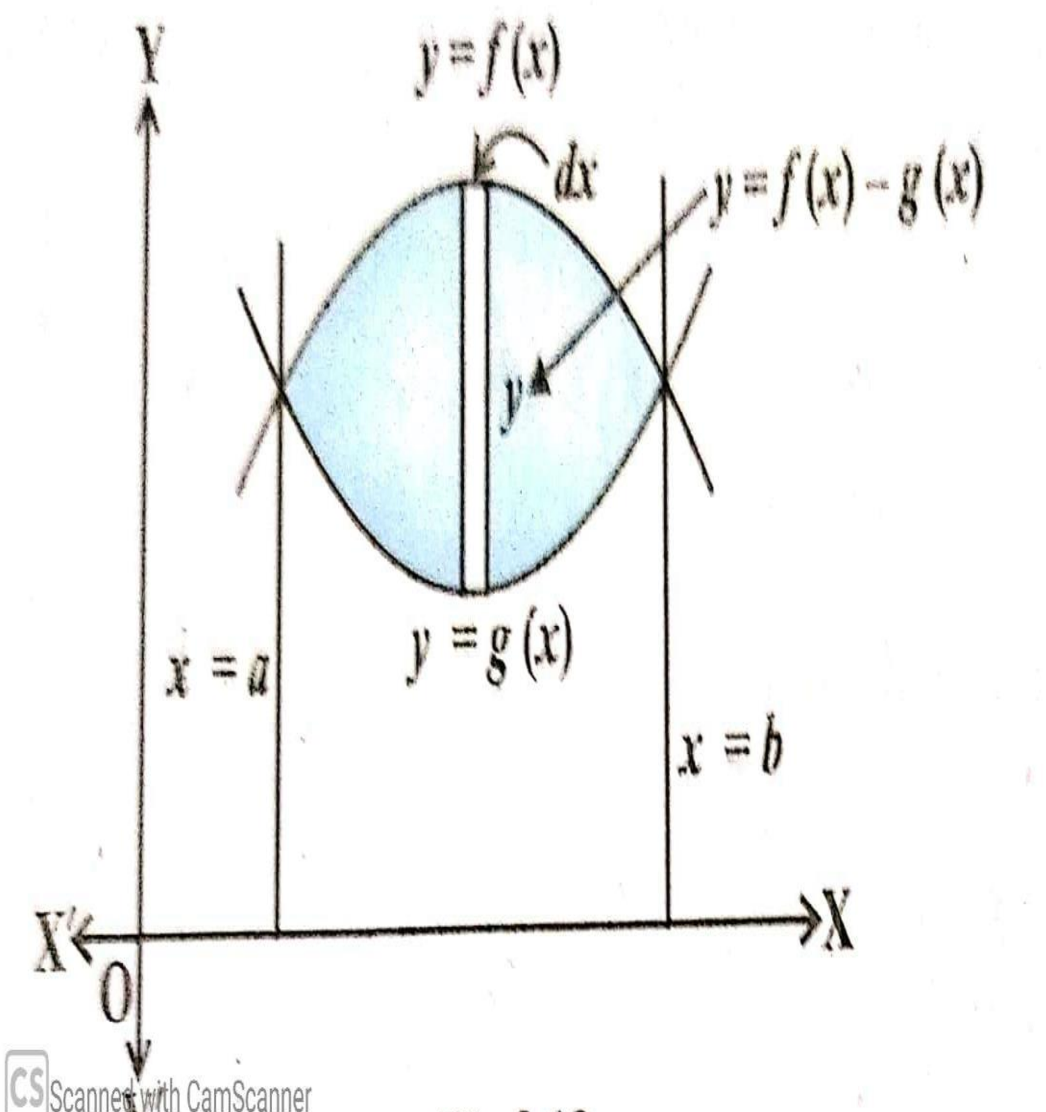
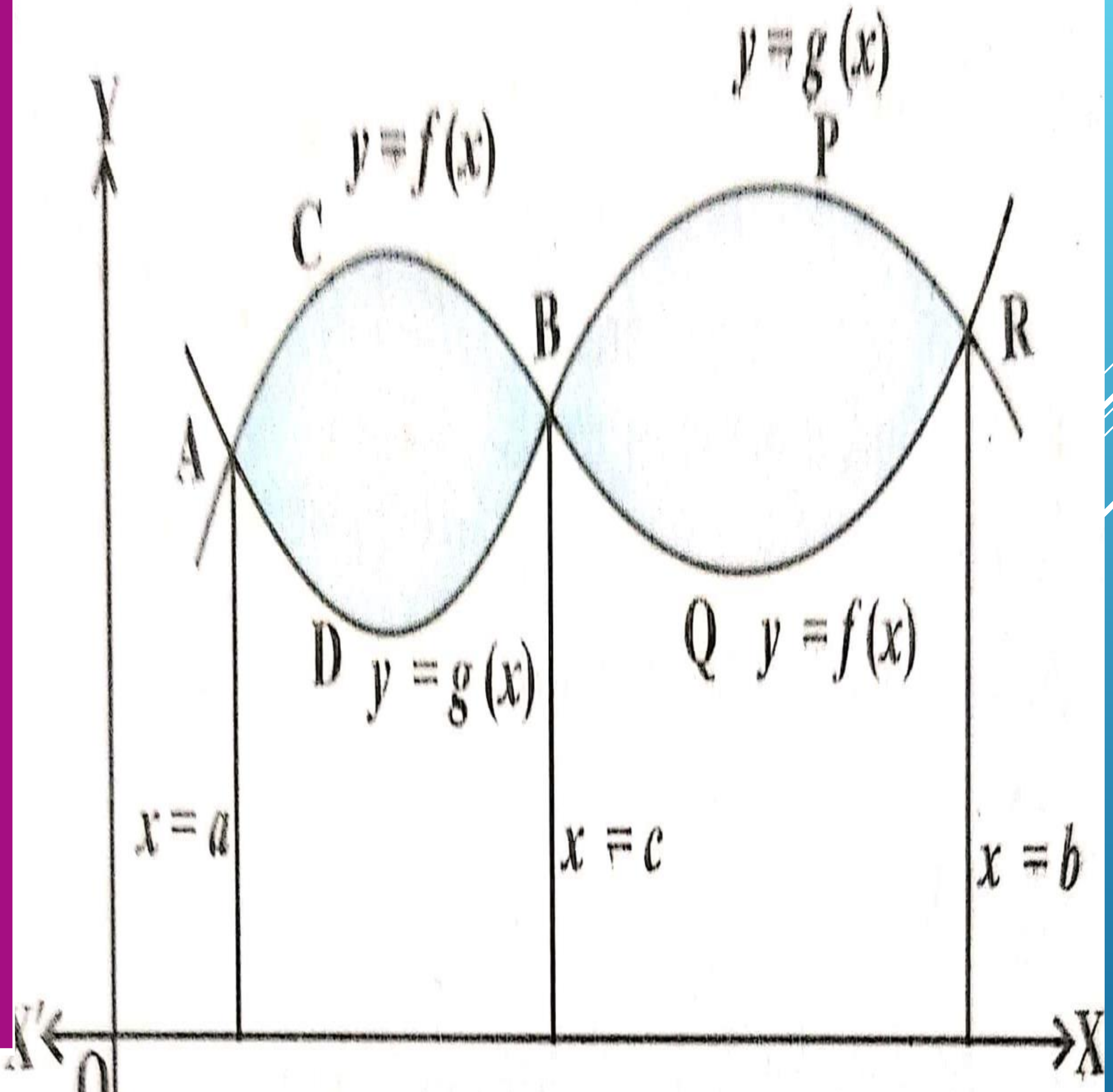


Fig 8.13

AREA BETWEEN TWO CURVES

If $f(x) \geq g(x)$ in $[a, c]$ and
$f(x) \leq g(x)$ in $[c, b]$, where
$a < c < b$ as shown in the side figure, then the area of the region bounded by the curves can be written as
Total area = Area of the region ACBDA + Area of the region BPRQB

$$= \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$$



EXAMPLE: 01. Find the area enclosed by the circle $x^2 + y^2 = a^2$

The whole area enclosed by the given circle
= 4 (area of the region AOBA bounded bythe curve, x- axis and the ordinates x = 0 and x = a)[as the circle is symmetrical about both x –axis and y-axis] = 4 $\int_0^a y dx$(taking vertical strips)

$$= 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

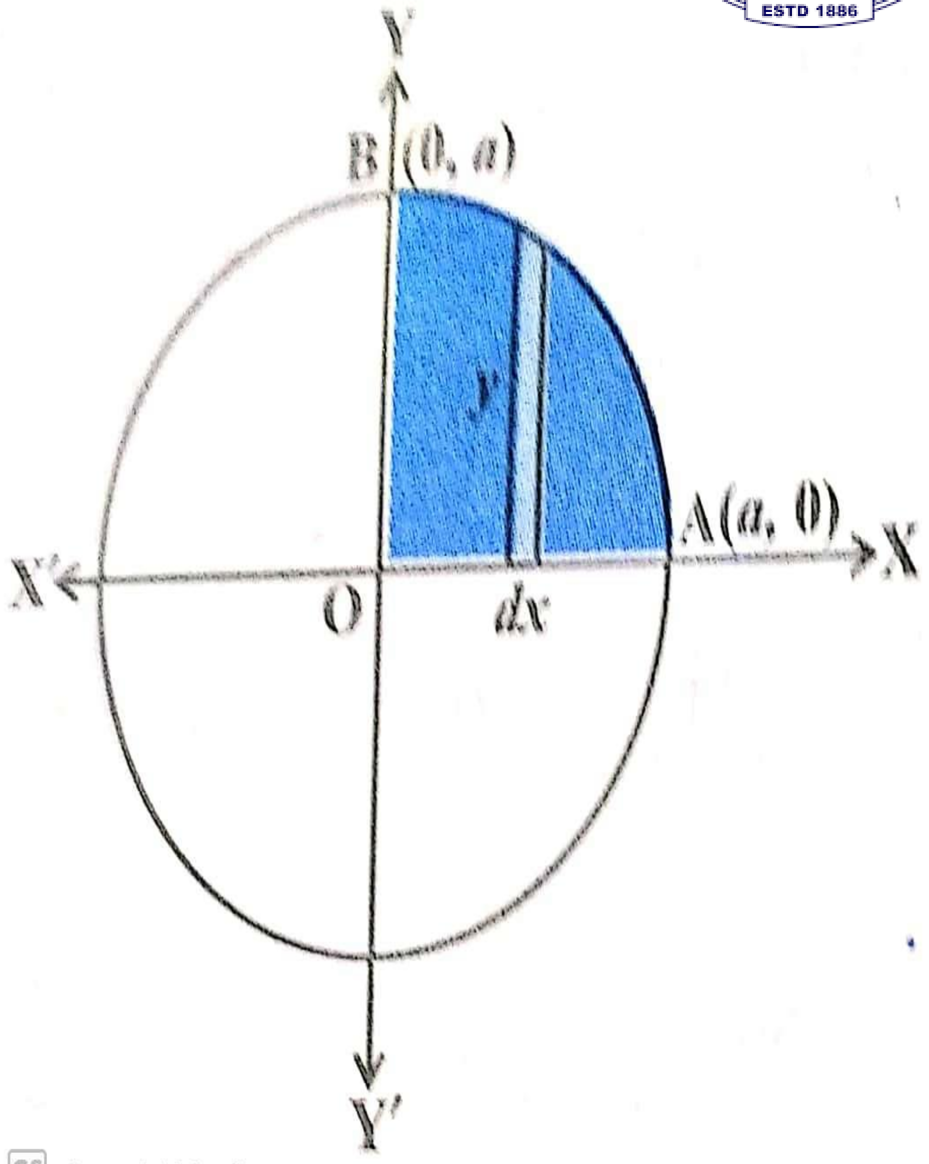
Since , $x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$

As the region AOBA lies in the first quadrant. Y is taken as positive. Integrating , we get the whole area enclosed by the given circle

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2} \times 0 - \frac{a^2}{2}\sin^{-1} 1 - 0\right)\right]$$

$$= 4\left(\frac{a^2}{2}\right)\left(\frac{\pi}{2}\right) = \pi a^2$$



ALTERNATIVE METHOD...

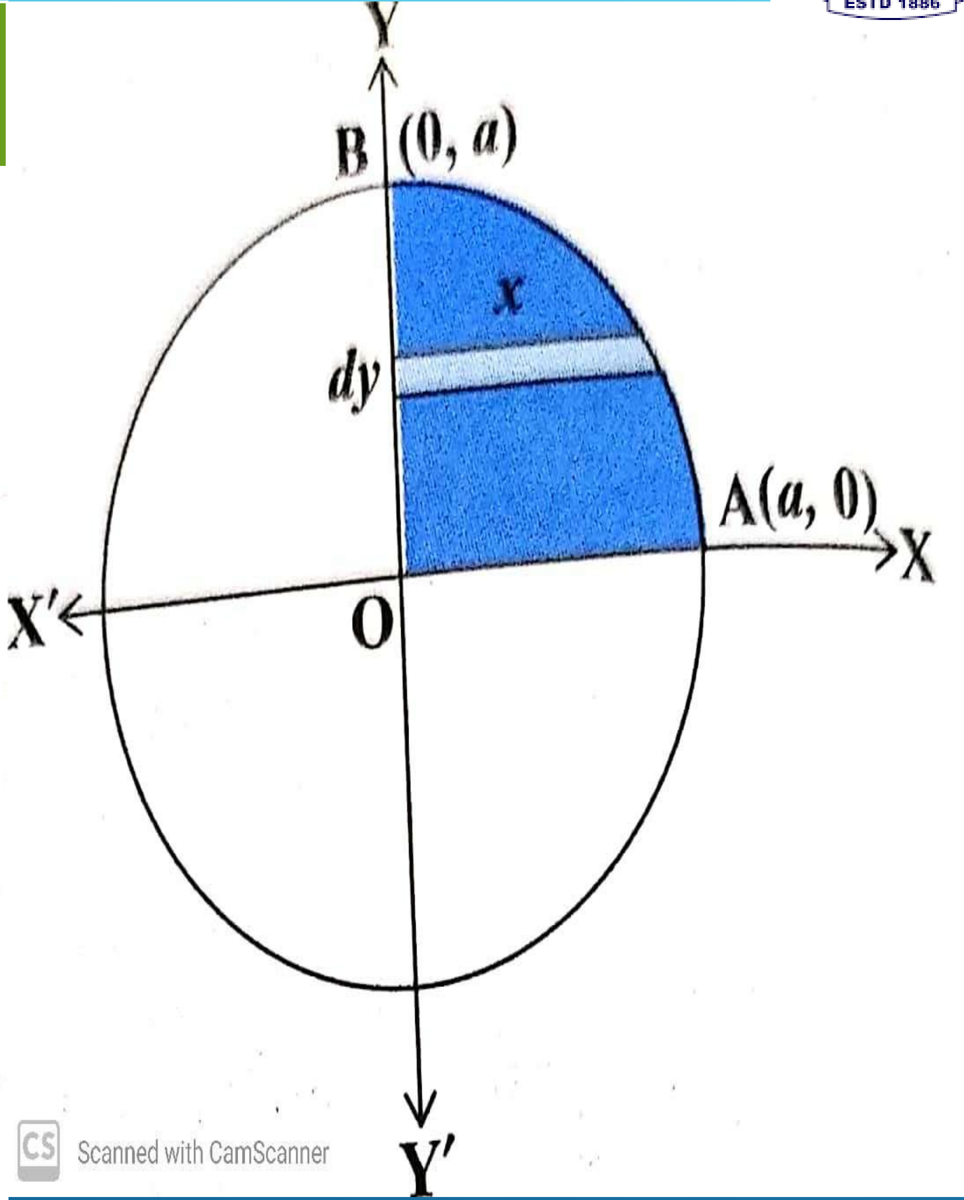
(Considering horizontal strips as shown in the side figure)
The whole area of the region enclosed by circle

$$= 4\int_0^a x\,dy = 4\int_0^a \sqrt{a^2 - y^2}\,dy \qquad \text{(Why?)}$$

$$= 4\left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= 4\frac{a^2}{2}\frac{\pi}{2} = \pi a^2$$



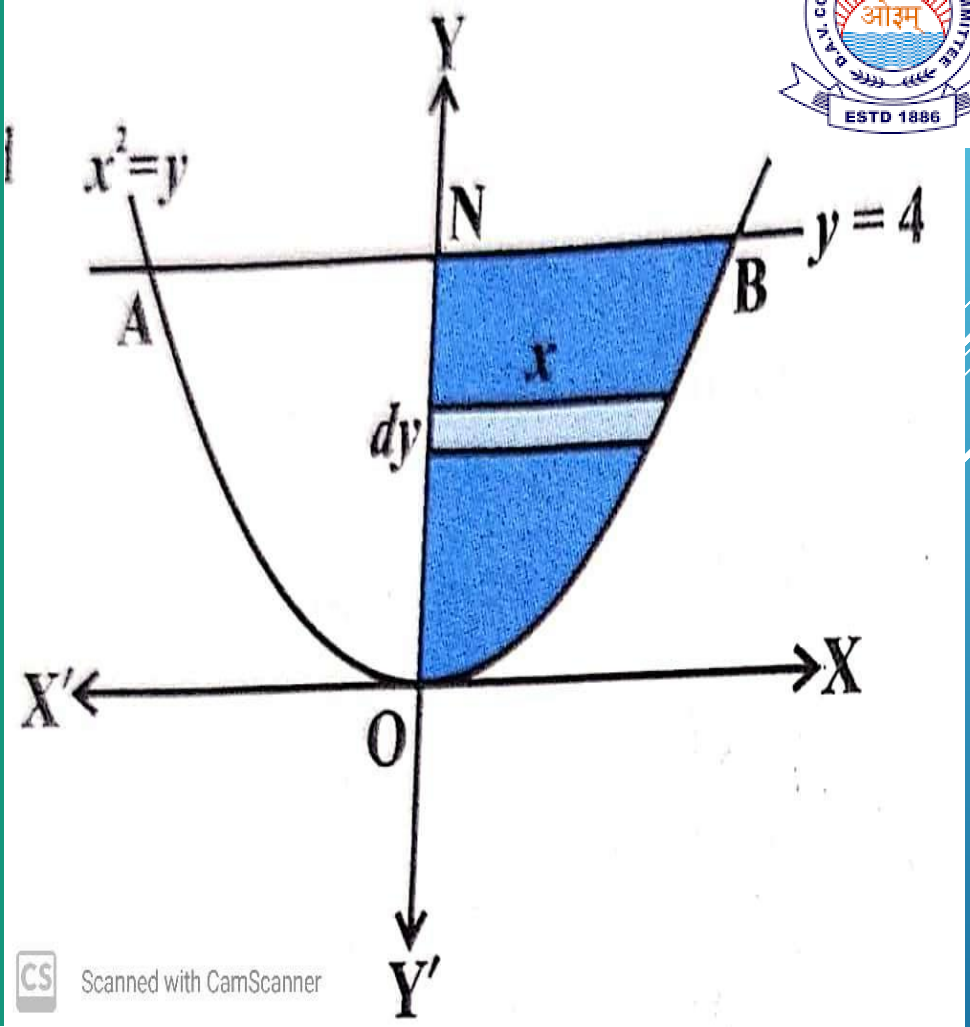
The area of the region bounded by a curve and a line.

Find the area of the region bounded by the curve $y = x^2 and\ the\ line y = 4$

Since the given curve represented by the equation $y = x^2$ is a Parabola symmetrical about y-axis only, therefore, from the side figure , the required area of the region AOBA is given by

= 2 $\int_0^4 x dy$ = 2(area of the region BONB bounded by curve, y-axis and the lines y = 0 and y = 4)

= 2 $\int_0^4 \sqrt{y}\, dy$ = $2 \times \frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^4$ = $\frac{4}{3} \times 8 = \frac{32}{3}$



The area of the region bounded by a curve and a line.

Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$

ALTERNATIVE METHOD

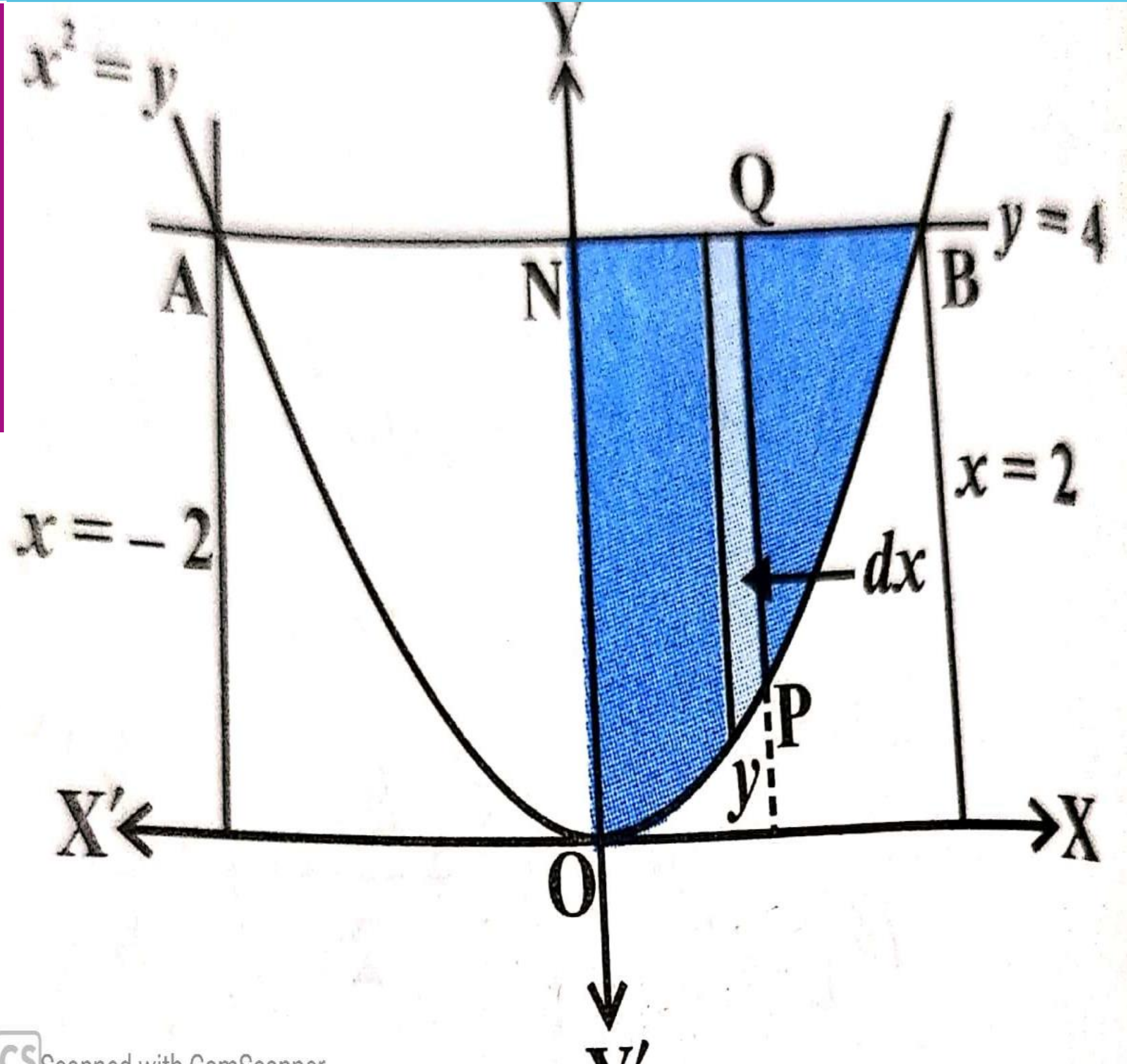
The region AOBA may be stated as the region bounded by the curve $y = x^2$, $y = 4$ and the ordinates $x = -2$ and $x = 2$.
Therefore, the area of the region AOBA

$$= \int_{-2}^{2} y\,dx$$

[y = (y-coordinate of Q) – (y-coordinate of P) = $4 - x^2$]

$$= 2\int_{0}^{2}\left(4 - x^2\right)dx \quad \text{(Why?)}$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = 2\left[4\times 2 - \frac{8}{3}\right] = \frac{32}{3}$$



EXAMPLE:03

Find the area of the region in the first quadrant enclosed by x-axis, the line $y = x$, and the circle $x^2 + y^2 = 32$

SOLN.

T the given equations are

$y = x$ ……….(1)

$x^2 + y^2 = 32$ ……..(2)

Solving equation(1) and (2), we get x=4 & y=4

i.e, circle and straight line intersect each other at pt B(4,4) in the first quadrant in the side fig.

Draw perpendicular BM to the x-axis.

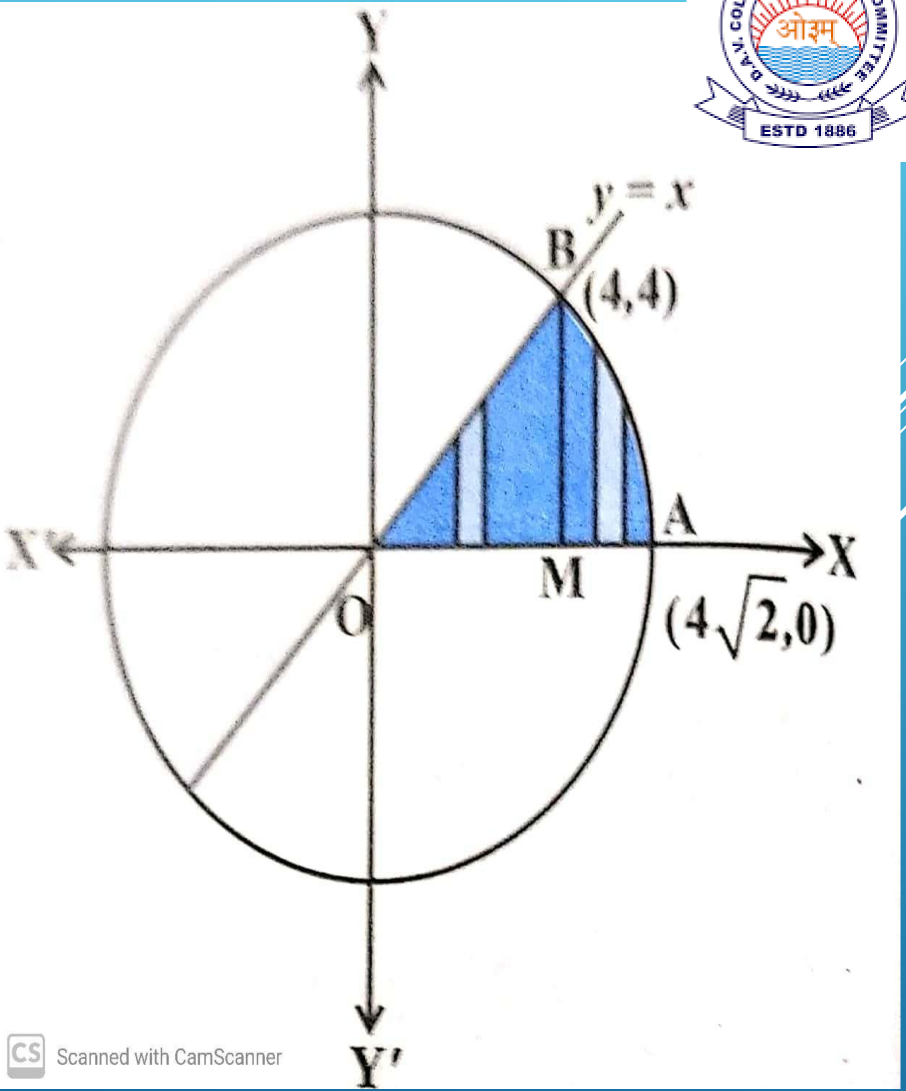
Therefore the required area

= area of the region OBMO+ area of the region BMAB.

Now, the area of the region OBMO

= $\int_0^4 y\,dx$ = $\int_0^4 x\,dx$ ……..(3)

=$\frac{1}{2}\left[x^2\right]_0^4$ = 8



Continue..

Again, the area of the region BMAB

$$= \int_{4}^{4\sqrt{2}} y\,dx = \int_{4}^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

$$= \left[\frac{1}{2}x\sqrt{32 - x^2} + \frac{1}{2} \times 32 \times \sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{1}{2}4\sqrt{2} \times 0 + \frac{1}{2} \times 32 \times \sin^{-1}1\right) - \left(\frac{4}{2}\sqrt{32 - 16} + \frac{1}{2} \times 32 \times \sin^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= 8\pi - (8 + 4\pi) = 4\pi - 8 \qquad \ldots (4)$$

Adding (3) and (4), we get, the required area = 4π.

Example : 4

Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1 and the ordinates $x = 0$ and $x = ae$, where, $b^2 = a^2(1-e^2)$ and $e < 1$

The required area of the region BOB'RFSB is enclosed by the ellipse and the lines $x = 0 \; and \; x = ae$.(from the side figure)

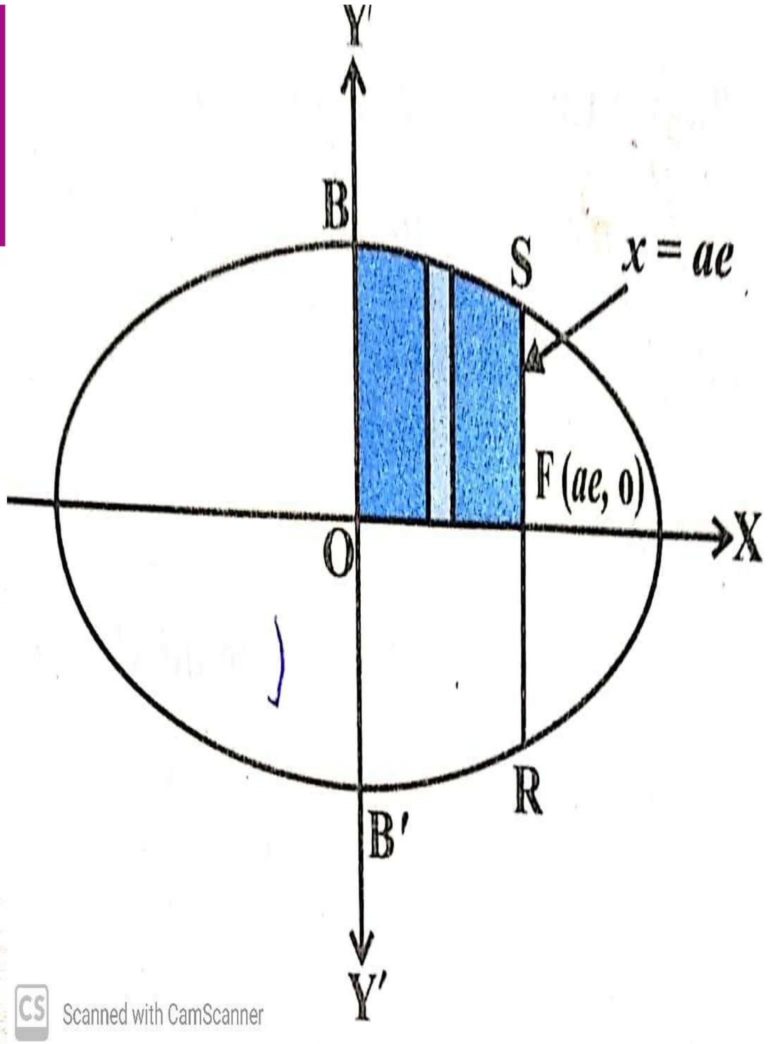
Hence, the area of the region BOB'RFSB

$$= 2\int_0^{ae} y\,dx = 2\frac{b}{a}\int_0^{ae}\sqrt{a^2-x^2}\,dx$$

$$= \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$$

$$= \frac{2b}{2a}\left[ae\sqrt{a^2-a^2e^2}+a^2\sin^{-1}e\right]$$

$$= ab\left[e\sqrt{1-e^2}+\sin^{-1}e\right]$$



Example-05

From the following figure, AOBA is the part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that OA = 2 and OB = 6. Find the area between the arc AB and the chord AB.

Solution Given equation of the ellipse $9x^2 + y^2 = 36$ can be expressed as $\frac{x^2}{4} + \frac{y^2}{36} = 1$ or $\frac{x^2}{2^2} + \frac{y^2}{6^2} = 1$ and hence, its shape is as given in Fig 8.17.

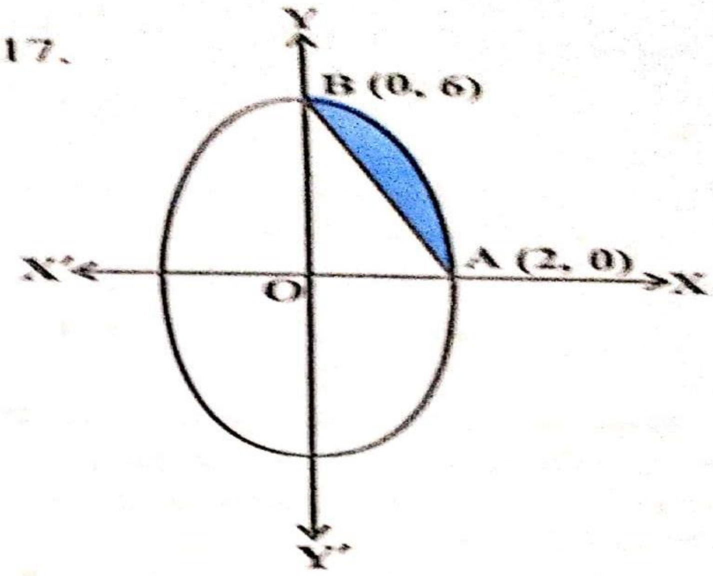


Fig 8.17

Accordingly, the equation of the chord AB is

$$y - 0 = \frac{6-0}{0-2}(x-2)$$

or $$y = -3(x-2)$$

or $$y = -3x + 6$$

Area of the shaded region as shown in the Fig 8.17.

$$= 3\int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (6-3x)\,dx \quad \text{(Why?)}$$

$$= 3\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[6x - \frac{3x^2}{2}\right]_0^2$$

$$= 3\left[\frac{2}{2}\times 0 + 2\sin^{-1}(1)\right] - \left[12 - \frac{12}{2}\right] = 3\times 2\times\frac{\pi}{2} - 6 = 3\pi - 6$$

For correct equation of line 1.5 marks
Correct figure:1.5 marks

2 marks

1 marks

Example:06

Using integration find the area of the region bounded by the triangle whose vertices are (1,0), (2,2) and (3,1).

Solution Let A (1, 0), B (2, 2) and C (3, 1) be the vertices of a triangle ABC (Fig 8.18).

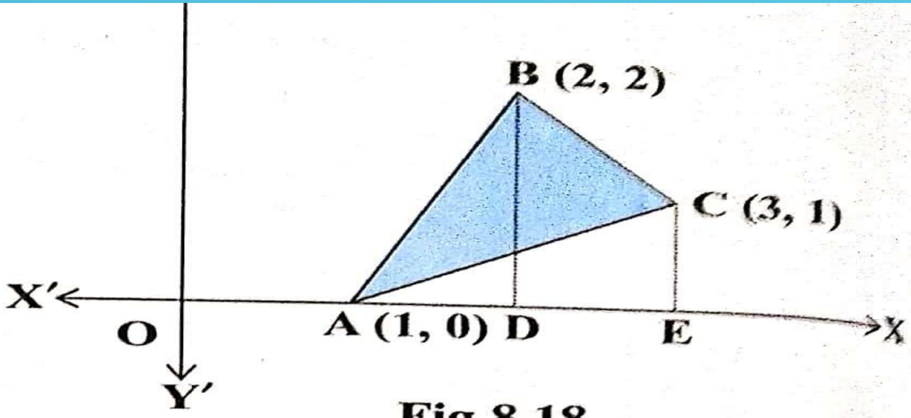


Fig 8.18

Area of ΔABC

= Area of ΔABD + Area of trapezium BDEC − Area of ΔAEC

Now equation of the sides AB, BC and CA are given by

$$y = 2(x-1),\ y = 4 - x,\ y = \frac{1}{2}(x-1), \text{ respectively.}$$

Hence, area of Δ ABC $= \int_1^2 2(x-1)\,dx + \int_2^3 (4-x)\,dx - \int_1^3 \frac{x-1}{2}\,dx$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2\left[\left(\frac{2^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right] + \left[\left(4\times3 - \frac{3^2}{2}\right) - \left(4\times2 - \frac{2^2}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{3^2}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{3}{2}$$

Figure: 1.5 marks

Equations of three lines: 1.5 marks

1.5 marks

1.5 marks

Example:07

Find the area of the region included between the parabola $y = \frac{3x^2}{4}$ and the line $3x - 2y + 12 = 0$

Solving the equations of the given curves $y = \frac{3x^2}{4}$ **and** **$3x - 2y + 12 = 0$, we get**

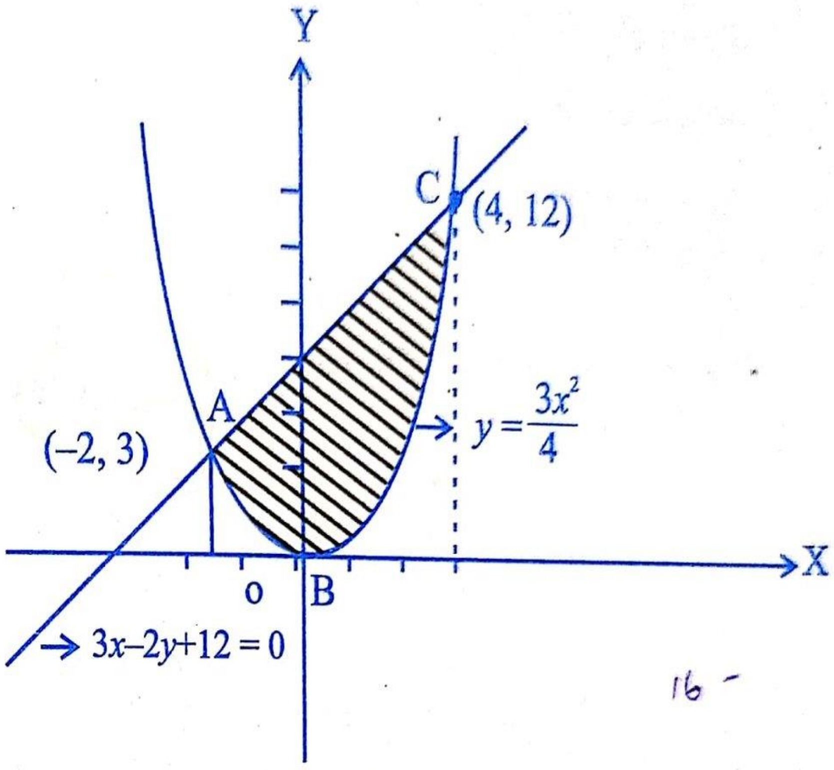
$3x^2 - 6x - 24 = 0 \Rightarrow (x-4)(x+2) = 0$

$\Rightarrow x = 4,\ x = -2$ which give $y = 12,\ y = 3$

From Fig.8.6, the required area = area of ABC

$$= \int_{-2}^{4}\left(\frac{12+3x}{2}\right)dx - \int_{-2}^{4}\frac{3x^2}{4}dx$$

$$= \left(6x + \frac{3x^2}{4}\right)_{-2}^{4} - \left[\frac{3x^3}{12}\right]_{-2}^{4} = 27 \text{ sq units.}$$



Solutions of x and y: 1.5 marks

For correct fig. 2 marks

2.5 marks

Example:08

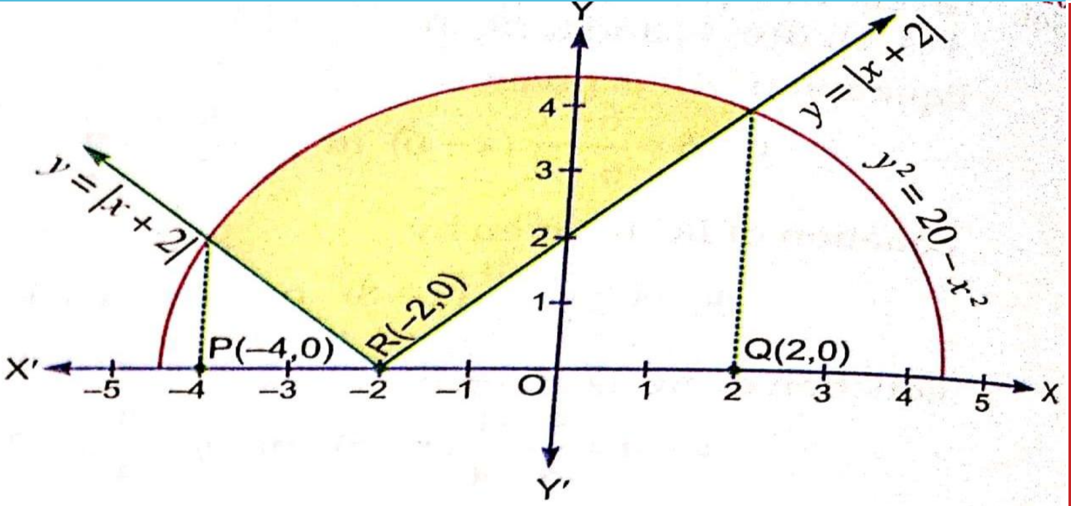
Using integration, Find the area of the following region:

$\{(x.y): |x+2| \le y \le \sqrt{20-x^2}\}$ CBSE-2010

Sol. Given region is $\{(x, y) : |x+2| \le y \le \sqrt{20-x^2}\}$.

It consists of inequalities $y \ge |x+2|$ and $y \le \sqrt{20-x^2}$

Plotting these inequalities, we obtain the adjoining shaded region.



Solving $y = x + 2$ and $y^2 = 20 - x^2$

$\Rightarrow$ $(x+2)^2 = 20 - x^2$

$\Rightarrow$ $2x^2 + 4x - 16 = 0$

or $(x+4)(x-2) = 0$

$\Rightarrow$ $x = -4, 2$

The required area $= \int_{-4}^{2} \sqrt{20-x^2}\,dx - \int_{-4}^{-2} -(x+2)\,dx - \int_{-2}^{2} (x+2)\,dx$

$$= \left[\frac{x}{2}\sqrt{20-x^2} + \frac{20}{2}\sin^{-1}\frac{x}{\sqrt{20}}\right]_{-4}^{2} + \left[\frac{x^2}{2} + 2x\right]_{-4}^{-2} - \left[\frac{x^2}{2} + 2x\right]_{-2}^{2}$$

$$= 4 + 10\sin^{-1}\frac{1}{\sqrt5} + 4 + 10\sin^{-1}\left(\frac{2}{\sqrt5}\right) + [2 - 4 - 8 + 8] - [2 + 4 - 2 + 4]$$

$$= 8 + 10\left(\sin^{-1}\frac{1}{\sqrt5} + \sin^{-1}\frac{2}{\sqrt5}\right) - 2 - 8 = -2 + 10\left(\sin^{-1}\frac{1}{\sqrt5} + \sin^{-1}\frac{2}{\sqrt5}\right)$$

$$= -2 + 10\sin^{-1}\left[\frac{1}{\sqrt5}\sqrt{1-\frac{4}{5}} + \frac{2}{\sqrt5}\sqrt{1-\frac{1}{5}}\right] = -2 + 10\sin^{-1}\left[\frac{1}{5} + \frac{4}{5}\right] = -2 + 10\sin^{-1}1$$

$$= -2 + 10\,\frac{\pi}{2} = (5\pi - 2) \text{ sq units.}$$

For the obtaining the values of x and y; 1.5 marks

Correct figre: 2 marks

2.5 marks

Example:09

Using integration find the area of the triangular region whose sides have equations $y = 2x + 1, y = 3 + 1$ and $x = 4$ (CBSE DELHI 2011)

Sol. The given lines are

$$y = 2x + 1 \quad \ldots(i)$$

$$y = 3x + 1 \quad \ldots(ii)$$

$$x = 4 \quad \ldots(iii)$$

For intersection point of (*i*) and (*iii*)

$$y = 2 \times 4 + 1 = 9$$

Coordinates of intersecting point of (*i*) and (*iii*) is (4, 9)

For intersection point of (*ii*) and (*iii*)

$$y = 3 \times 4 + 1 = 13$$

i.e., Coordinates of intersection point of (*ii*) and (*iii*) is (4, 13)

For intersection point of (*i*) and (*ii*)

$$2x + 1 = 3x + 1 \quad \Rightarrow \quad x = 0$$

$$\therefore \quad y = 1$$

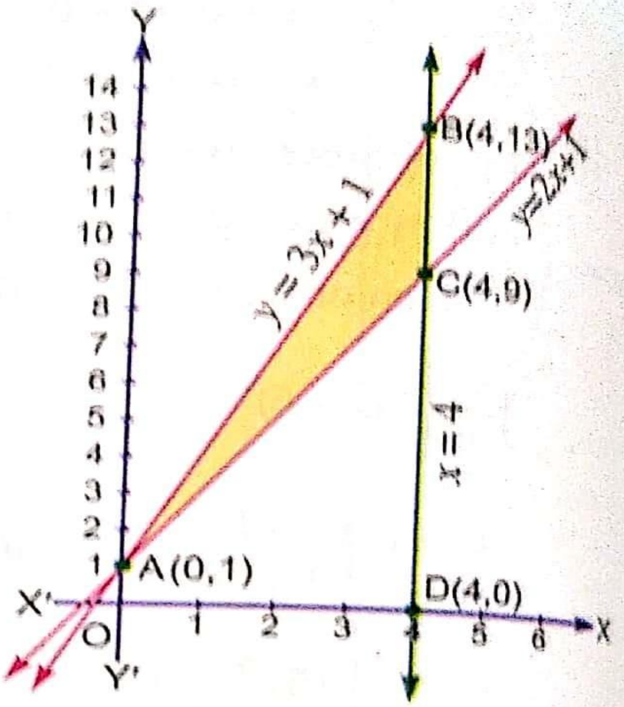
i.e., Coordinates of intersection point of (*i*) and (*ii*) is (0, 1).

Shaded region is required triangular region.

$\therefore$ Required Area = Area of trapezium $OABD$ – Area of trapezium $OACD$

$$= \int_0^4 (3x + 1)\, dx - \int_0^4 (2x + 1)\, dx = \left[3\frac{x^2}{2} + x\right]_0^4 - \left[\frac{2x^2}{2} + x\right]_0^4$$

$$= [(24 + 4) - 0] - [(16 + 4) - 0] = 28 - 20 = 8 \text{ sq units.}$$



For Fig. 2 marks

For three correct vertices 1.5 marks

1.5 marks

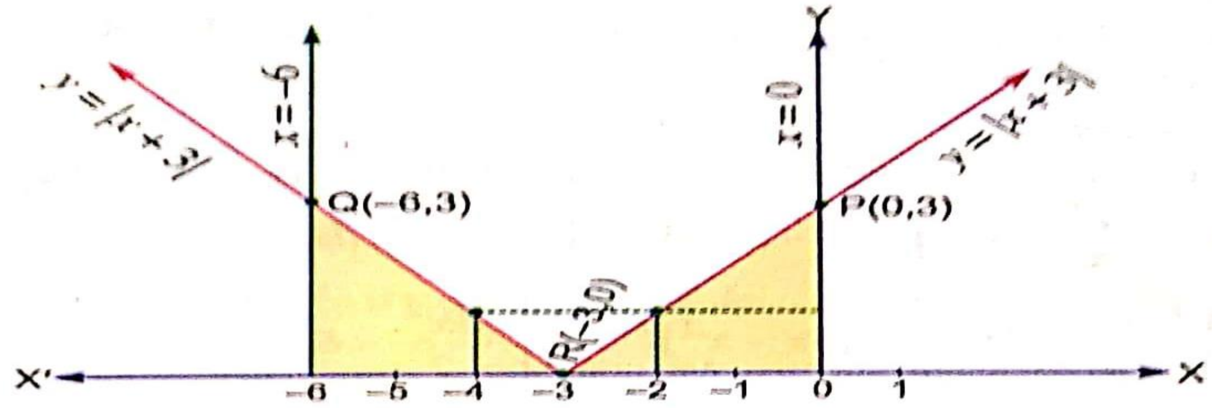
1 mark

Example:10

Sketch the graph of $y = |x+3|$ and evaluate the area under the curve $y = |x+3|$ above x-axis and between $x = -6$ to $x = 0$. [CBSE(AI)-2011]

Sol. For graph of $y = |x+3|$

x	0	−3	−6	−2	−4
y	3	0	3	1	1



Shaded region is the required region.

Hence, Required area $= \int_{-6}^{0} |x+3|\,dx$

$= \int_{-6}^{-3} |x+3|\,dx + \int_{-3}^{0} |x+3|\,dx$ [By Property of definite integral]

$= \int_{-6}^{-3} -(x+3)\,dx + \int_{-3}^{0} (x+3)\,dx$ $\left[\begin{array}{l} x+3 \geq 0 \text{ if } -3 \leq x \leq 0 \\ x+3 \leq 0 \text{ if } -6 \leq x \leq -3 \end{array}\right]$

$= -\left[\frac{x^2}{2} + 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0} = -\left[\left(\frac{9}{2} - 9\right) - \left(\frac{36}{2} - 18\right)\right] + \left[0 - \left(\frac{9}{2} - 9\right)\right]$

$= \frac{9}{2} + \frac{9}{2} = 9$ sq units.

For correct sketch of graph 3 marks

2 marks

1 mark

Example:11

Find the area of the region bounded by the parabola $y^2 = 2x$ and the line $x - y = 4$. [CBSE (F) 20013]

Sol. Given curves are $y^2 = 2x$ ....(*i*)

and $x - y = 4$ ....(*ii*)

Obviously, curve (*i*) is right handed parabola having vertex at (0, 0) and axis along +ve direction of *x*-axis while curve (*ii*) is a straight line.

For intersection point of curve (*i*) and (*ii*)

$$(x-4)^2 = 2x$$

$\Rightarrow \quad x^2 - 8x + 16 = 2x \quad \Rightarrow \quad x^2 - 10x + 16 = 0$

$\Rightarrow \quad x^2 - 8x - 2x + 16 = 0 \quad \Rightarrow \quad x(x-8) - 2(x-8) = 0$

$\Rightarrow \quad (x-8)(x-2) = 0 \quad \Rightarrow \quad x = 2, 8$

$\Rightarrow \quad y = -2, 4$

Intersection points are (2, –2), (8, 4)

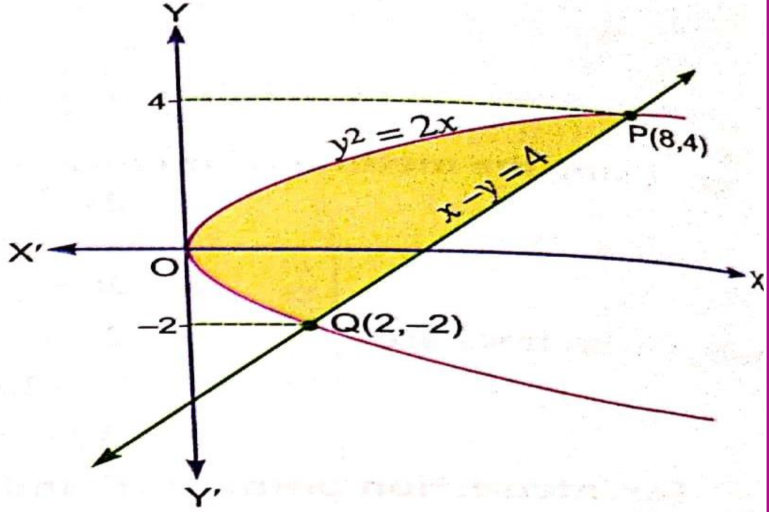
Therefore, required Area = Area of shaded region

$$= \int_{-2}^{4} (y+4)\,dy - \int_{-2}^{4} \frac{y^2}{2}\,dy$$

$$= \left[\frac{(y+4)^2}{2}\right]_{-2}^{4} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{2}\cdot[64-4] - \frac{1}{6}[64+8]$$

$$= 30 - \frac{72}{6} = 18 \text{ sq units.}$$



For correct soln of line and parabola. 2 marks
For figure; 2 marks

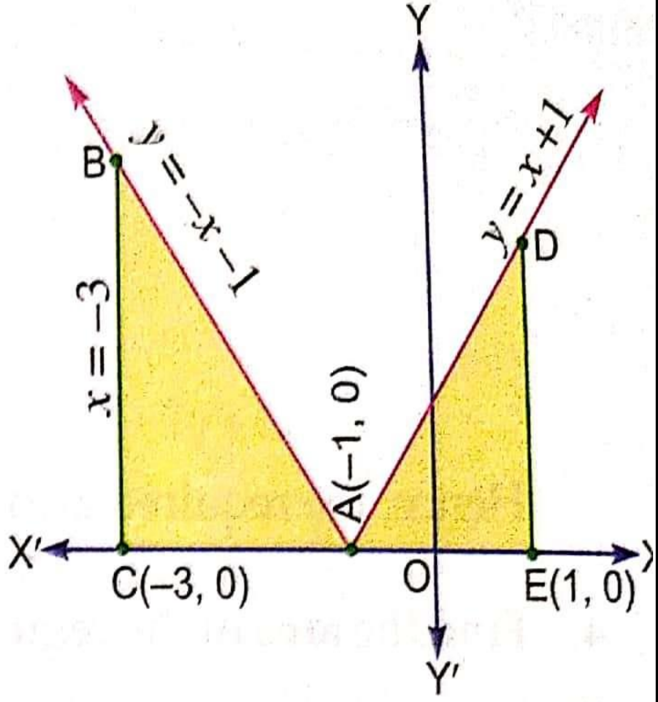
2 marks

Example12

Sketch the graph $y = |x+1,|$. Evaluate $\int_{-3}^{1} |x+1| dx$.What does this value represent on the graph?(HOTS)

Sol. We have, $y = |x+1| = \begin{cases} x+1, \text{ if } x+1 \ge 0 & i.e., \quad x \ge -1 \\ -(x+1), \text{ if } x+1 < 0 & i.e., \quad x < -1 \end{cases}$

So, we have $y = x+1$ for $x \ge -1$ and $y = -x-1$ for $x < -1$. Clearly, $y = x+1$ is a straight line cutting x and y-axes at $(-1, 0)$ and $(0, 1)$ respectively. So, $y = x+1, x \ge -1$ represents that portion of the line which lies on the right side of $x = -1$. Similarly, $y = -x-1, x < -1$ represents that part of the line $y = -x-1$ which is on the left side of $x = -1$. A rough sketch of $y = |x+1|$ is shown in fig.



Now, $\int_{-3}^{1} |x+1| \, dx = \int_{-3}^{-1} -(x+1) \, dx + \int_{-1}^{1} (x+1) \, dx$

$$= -\left[\frac{(x+1)^2}{2}\right]_{-3}^{-1} + \left[\frac{(x+1)^2}{2}\right]_{-1}^{1} = -\left[0 - \frac{4}{2}\right] + \left[\frac{4}{2} - 0\right] = 4 \text{ sq units}$$

This value represents the area of the shaded portion shown in figure.

Defining modulus function and getting correct limits : 2 marks

Correct figure: 1.5 marks

2.5 mars

Example:13

Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$

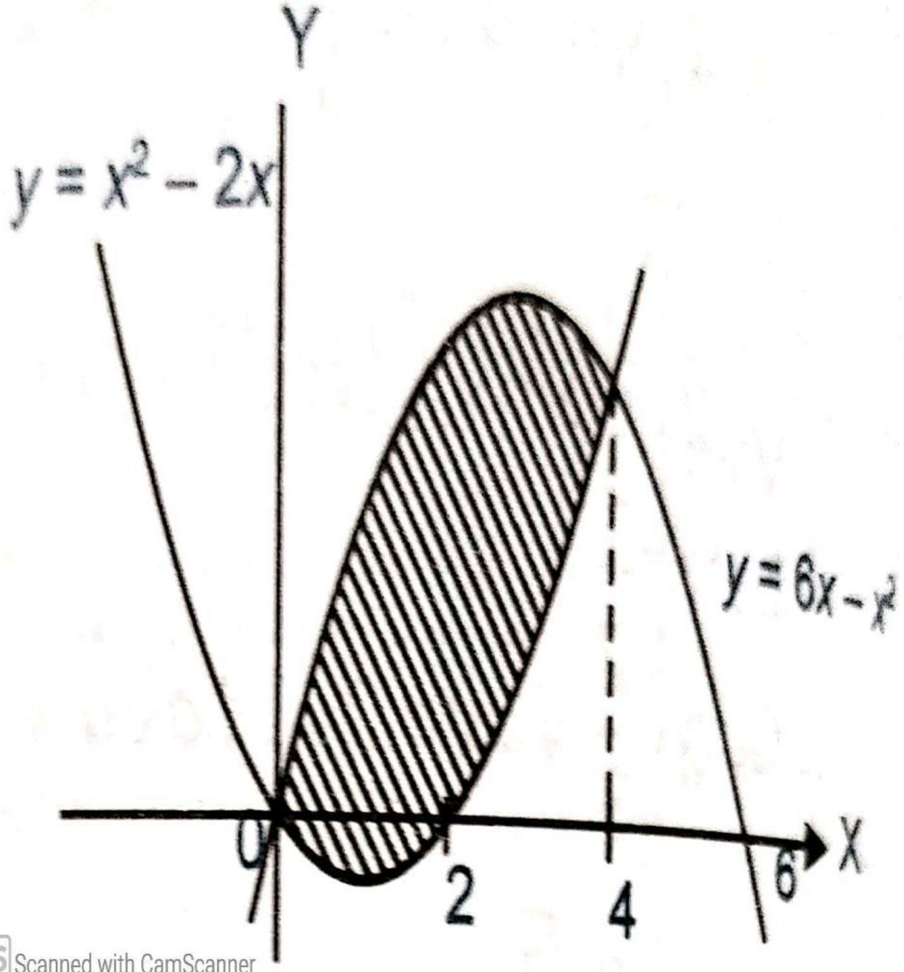
Curves are $y = 6x - x^2$ and $y = x^2 - 2x$

Curves intersect at (0, 0) and (4, 8)

$$\text{Area} = \int_0^4 \{(6x - x^2) - (x^2 - 2x)\}\, dx$$

$$\text{Area} = \int_0^4 (6x - x^2 - x^2 + 2x)dx = \int_0^4 (8x - 2x^2)dx$$

$$= 4x^2 - \frac{2x^3}{3}\Bigg]_0^4 = 64 - \frac{128}{3} = \frac{64}{3} \text{ sq units}$$



For correct solution of two curves: 1.5 marks
For correct figure: 2marks

Correct solutions: 2.5 marks

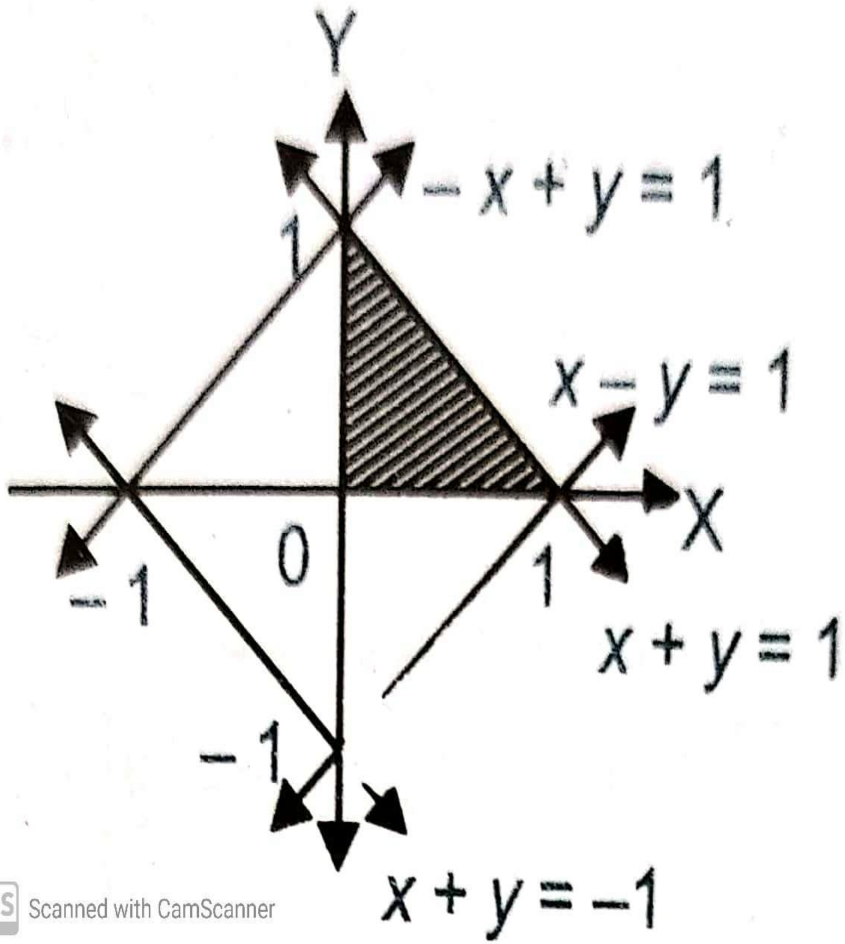
Example:14. **Using the method of integration find the area bounded by the curve $|x| + |y| = 1$.**

Here the shaded triangle is symmetrical to other three unshaded tringles

Curves are $x + y = 1$, $x + y = -1$, $-x + y = 1$, $x - y = 1$

Area = 4 x area in 1st quadrant.

$$= 4\int_0^1 (1 - x)\,dx = 4\left(x - \frac{x^2}{2}\right)_0^1 = 4\left(1 - \frac{1}{2}\right) = 2 \text{ sq units}$$



For correct four equations of lines: 2 Marks

For correct figure: 2 marks

2 marks

Activity: To evaluate the definite integral $\int_a^b \sqrt{1-x^2}\,dx$ as the limit of a sum and verify it by actual integration.

Pre-requisite knowledge : Knowledge of integration and geometry.

Materials required : Cardboard, white paper and graph paper.

Procedure :

1. Take a cardboard of a suitable size and paste a white paper on it.
2. Draw two lines perpendicular to each other, representing co-ordinate axes X'OX and YOY'.
3. Draw a quadrant of a circle with O as centre and radius 1 unit (10 cm), as shown in the figure.

   The curve in the 1st quadrant represents the graph of the function $\sqrt{1-x^2}$ in the closed interval [0, 1]

4. Let origin O be denoted by A_0 and the points where the curve meets the x-axis and y-axis be denoted by A_{10} and B_0 respectively.

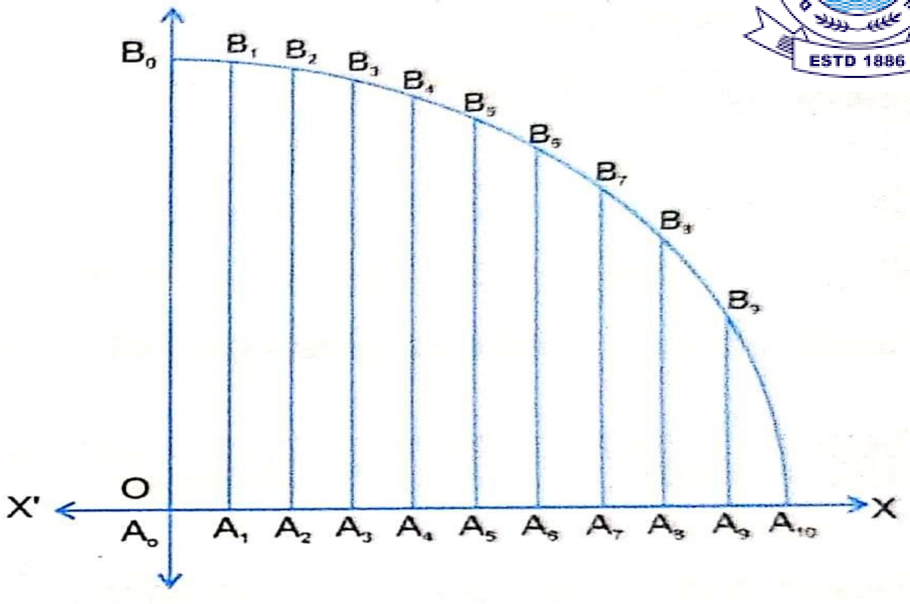


Fig. 19.1

5. Divide A_0A_{10} into 10 equal parts with points of division as, A_1, A_2, A_3, ..., A_9.
6. From each of the points A_1, A_2, A_3 .... A_9, draw perpendicular on the x-axis to meet the curve at the points B_1, B_2, B_3 ......, B_9. Measure the lengths of A_0B_0, A_1B_1, A_2B_2, ... A_9B_9 and call them as y_0, y_1, y_2, y_3 .... y_9, whereas width of each part, $A_0A_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5 = A_5A_6 = A_6A_7 = A_7A_8 = A_8A_9 = A_9A_{10} = 0.1$ unit.

Continue...

Observations

1. $y_0 = A_0B_0 = 1$ unit
   $y_1 = A_1B_1 = 0.99$ units
   $y_2 = A_2B_2 = 0.97$ units
   $y_3 = A_3B_3 = 0.95$ units
   $y_4 = A_4B_4 = 0.92$ units
   $y_5 = A_5B_5 = 0.87$ units
   $y_6 = A_6B_6 = 0.8$ units
   $y_7 = A_7B_7 = 0.71$ units
   $y_8 = A_8B_8 = 0.6$ units
   $y_9 = A_9B_9 = 0.43$ units
   $y_{10} = A_{10}B_{10}$ = which is very small and assumed to be 0.
2. Area of the quadrant of the circle
   = Area bounded by the curve and the two axes
   = sum of the areas of 10 trapeziums.
   $= \frac{1}{2} \times (0.1) \times [(1 + 0.99) + (0.99 + 0.97) + (0.97 + 0.95) + (0.95 + 0.92) + (0.92 + 0.87) + (0.87 + 0.8) + (0.8 + 0.71) + (0.71 + 0.6) + (0.6 + 0.43) + (0.43)]$
   $= (0.1) \times [0.5 + 0.99 + 0.97 + 0.95 + 0.92 + 0.87 + 0.80 + 0.71 + 0.60 + 0.43]$
   $= 0.1 \times 7.74 = 0.774$ sq. units (approximately)
3. Also, $\int_0^1 \sqrt{1-x^2}\,dx = \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{1}{2} \times \frac{\pi}{2} = \frac{3.14}{4} = 0.785$ sq. units.
4. The area of the quadrant as a limit of a sum is nearly the same as the area obtained by actual integration.

Conclusion

From the above activity we see that the definite integral $\int_a^b \sqrt{1-x^2}\,dx$ can be evaluated as the limit of a sum.

Application : Useful to give concept clarity of area bounded by the curves.

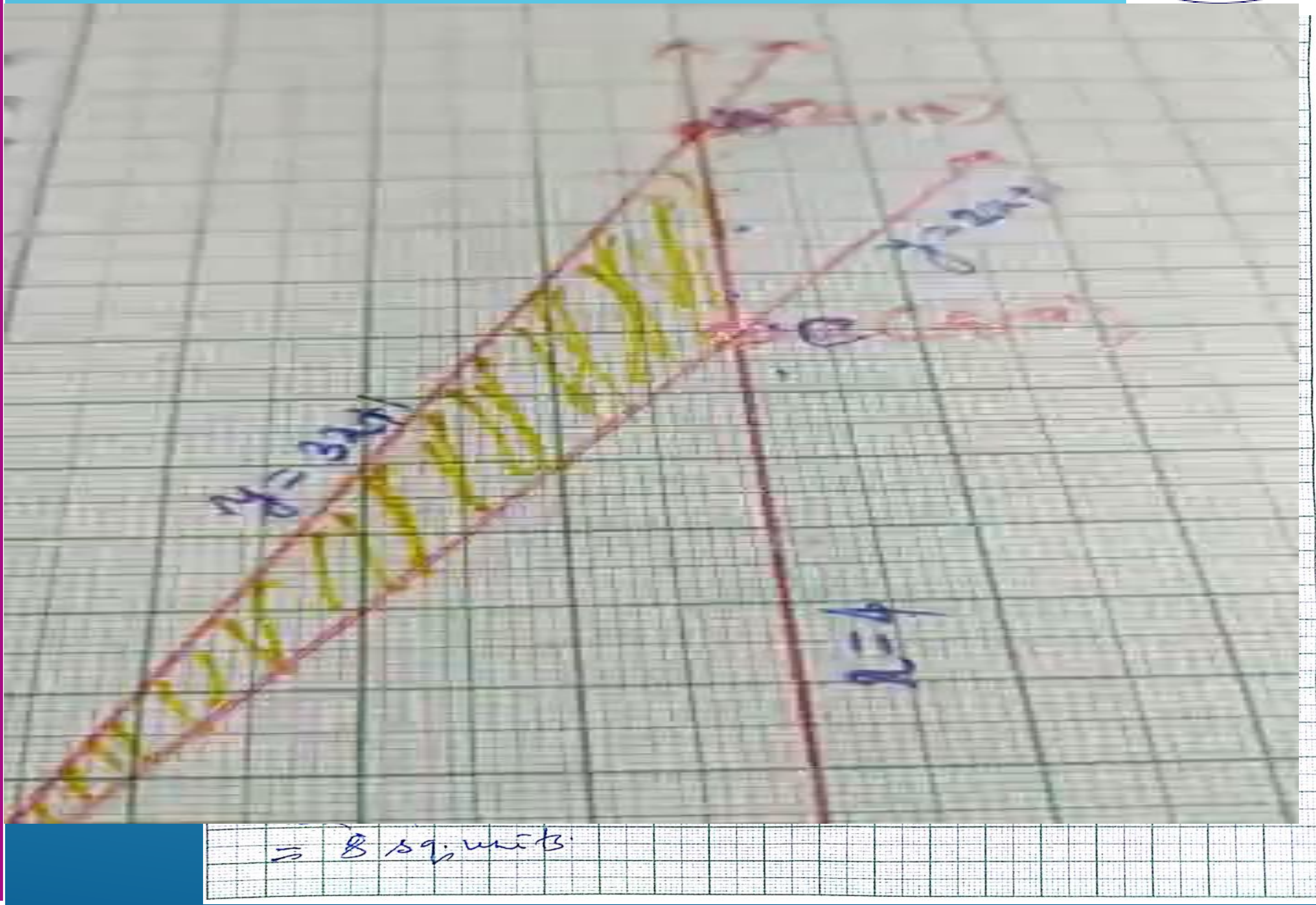
ACTIVITY: To test the area obtained by integration method and by using formula

Using integration find the area of the triangular region whose sides have equations

$y = 2x + 1,$

$y = 3x + 1 \; and \; x = 4$

(CBSE DELHI 2011)

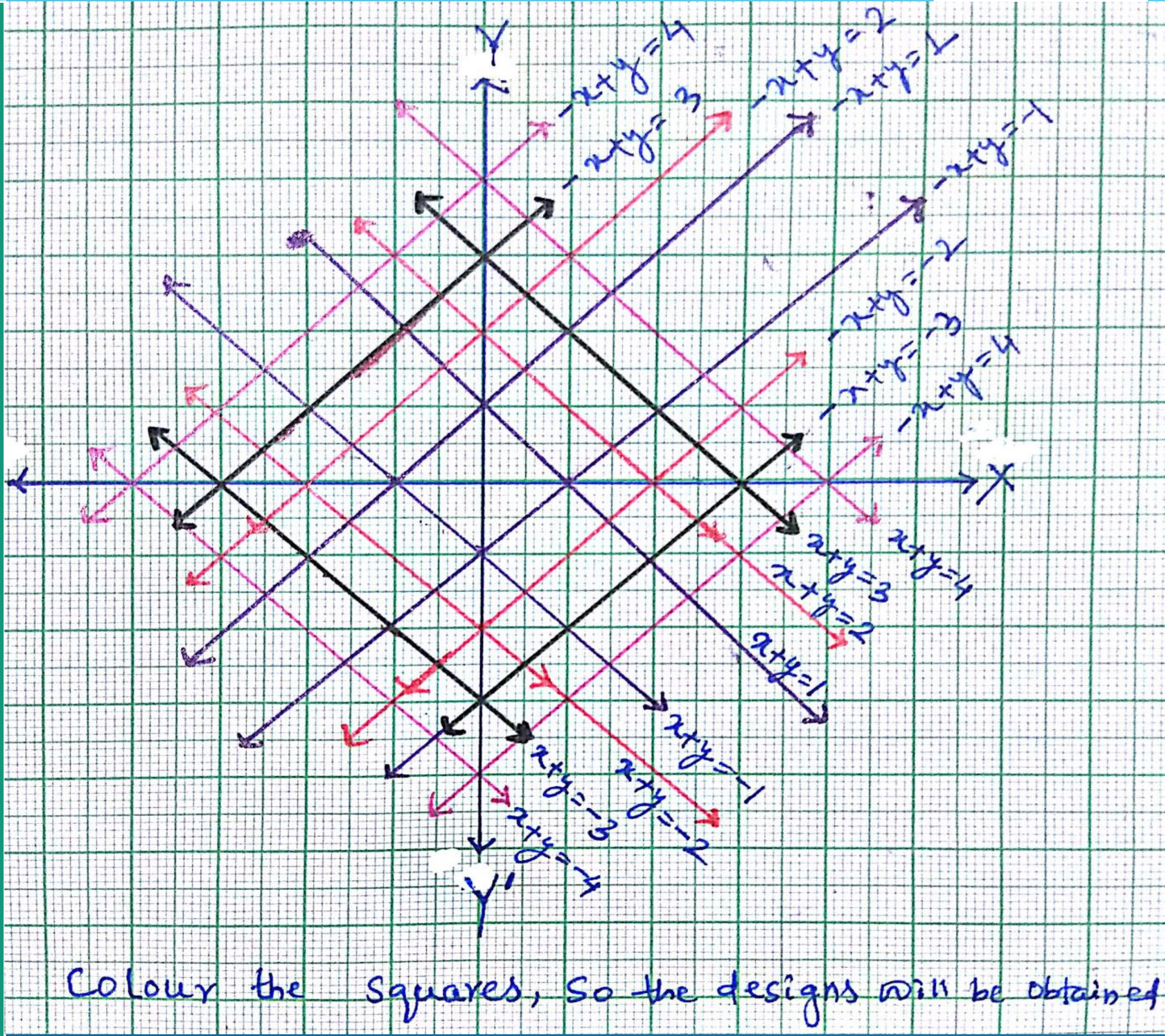


VIDEO LINK

ART INTEGRATION..

- Sketching the graph of **the curve**

$$|x| + |y| = n, n \in N$$



SOME ADDITIONAL PROBLEMS

01. Sketch the graph of $y = |x + 3|$ and evaluate the area under the curve $y = |x + 3|$ above x-axis and between $x = -6 \; to \; x = 0$.

02. using the method of integration, find the area of the region bounded by the lines 3x-2y+1=0,2x+3y-21=0 and x-5y+9=0

03. Find the area of the region included between the parabolas $y^2 = 4ax \; and \; x^2 = 4ay, where \; a > 0$

04. Sketch the graph $y = |x - 1|$. evaluate $\int_{-2}^{4} |x - 1| dx$. what does the value of this integral represent on the graph?

KEY TAKEAWAYS

KEY POINTS

- Area is a quantity that expresses the extent of a two-dimensional surface or shape, or planar lamina, in the plane.
- The area between the graphs of two functions is equal to the integral of one function, $f(x)$, minus the integral of the other function, $g(x)$: A = $\int_a^b [f(x) - g(x)]dx$, where $f(x)$ is the curve with the greater y-value.
- The area between a positive-valued curve and the horizontal axis, measured between two values, a and b (where $b>a$), on the horizontal axis, is given by the integral from a to b of the function that represents the curve: A= $\int_a^b f(x)\, dx$.

KEY TERMS

AREA: A measure of the extent of a surface measured in square units

CURVE: A simple figure containing no straight portions and no angles

AXIS: A fixed, one-dimensional figure, such as a line or arc, with an origin and orientation and such that its points are in one-to-one correspondence with a set of numbers; an axis forms part of the basis of a space or is used to position and locate data in a graph (a coordinate axis)

Mind Mapping.....

Area bounded by a curve, ordinates and x-axis

- **Curve, parabola, ellipse and ordinates**

Area bounded by a curve, abscissa and y-axis

- **Curves, parabola, ellipse, circles etc**

Area between two curves

- **Two circles, circle and parabola, two parabola, two ellipse etc**

THANK YOU