Injectivity

Ex-1:

Prove that $f: R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$ is one-one .

Proof:

Let $x, y \in R_+$ and $f(x) = f(y)$

$\Rightarrow 9x^2 + 6x - 5 = 9y^2 + 6y - 5$

$\Rightarrow 9(x^2 - y^2) + 6(x - y) = 0$

$\Rightarrow (x - y)(9(x + y) + 6) = 0$

$\Rightarrow x - y = 0 \text{ } or \text{ } 9(x + y) + 6 = 0$

If $9(x + y) + 6 = 0$, then $x = -\left(\frac{2}{3} + y\right)$

As $y \in R_+ \Rightarrow -\left(\frac{2}{3} + y\right) \in R_- \Rightarrow x \in R_-$

Which is not possible as we have considered initially $\in R_+$.

So only possibility is $x - y = 0 \text{ } i.e. x = y$.

Ex:2

Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$ is one-one .

Proof:

Let us redefine the function $f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x} & x \geq 0 \\ \frac{x}{1-x} & x < 0 \end{cases}$.

We have to choose two real numbers from the domain.

There are four possibilities

(i) $x, y \in R_+$ (ii) $x, y \in R_-$ (iii) $x \in R_+ , y \in R_-$ (iv) $x \in R_- , y \in R_+$

Case-1:

Let $x, y \in R_+$ and $f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$.

Case-2:

Let $x, y \in R_-$ and $f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - xy \Rightarrow x = y$.

Case-3:

Let $\in R_+ , y \in R_-$. So $x \neq y$

Now $f(x) = \frac{x}{1+x} > 0$ and $f(y) = \frac{y}{1-y} < 0 \Rightarrow f(x) \neq f(y)$

Case-4:

Let $\in R_- , y \in R_+$. So $x \neq y$

Now $f(x) = \frac{x}{1-x} < 0$ and $f(y) = \frac{y}{1+y} > 0 \Rightarrow f(x) \neq f(y)$

Hence the function is one-one .

Surjectivity

Ex:1

Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$ is onto .

Proof:

Let us redefine the function $f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x} & x \geq 0 \\ \frac{x}{1-x} & x < 0 \end{cases}$.

For ≥ 0 , $f(x) = \frac{x}{1+x} \geq 0$. For $x < 0$, $f(x) = \frac{x}{1-x} < 0$

Case-1

Let $y \in (-1,0)$. Then $y = \frac{x}{1-x}$ for some x $\Rightarrow x = y - xy \Rightarrow x = \frac{y}{1+y} < 0$

Now $f(x) = f\left(\frac{y}{1+y}\right) = \frac{\frac{y}{1+y}}{1-\frac{y}{1+y}} = y$.

So for $y \in (-1,0)$ there exists $x \in R$ such that $f(x) = y$

Case-2

Let $y \in [0,1)$. Then $y = \frac{x}{1+x}$ for some x $\Rightarrow x = y + xy \Rightarrow x = \frac{y}{1-y} \geq 0$

Now $f(x) = f\left(\frac{y}{1-y}\right) = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = y$.

So for $y \in [0,1)$ there exists $x \in R$ such that $f(x) = y$

Hence the function is onto .

Ex-2:

Prove that $f: R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$ is onto .

Method-1:

$f(x) = 9x^2 + 6x - 5 = (3x+1)^2 - 6$

As $x \in R_+ \Rightarrow x \geq 0 \Rightarrow 3x + 1 \geq 1 \Rightarrow (3x+1)^2 - 6 \geq -5 \Rightarrow f(x) \geq -5$

So $R_f = [-5, \infty)$ =Co-domain $\Rightarrow$ Function is onto .

Method-2:

Let $y \in [-5, \infty)$ *and* $y = 9x^2 + 6x - 5 \Rightarrow 9x^2 + 6x - (5 + y) = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{y+6}}{3}$

$x = \frac{-1-\sqrt{y+6}}{3} \notin R_+$. So discarded.

But $x = \frac{-1+\sqrt{y+6}}{3} \in R_+$ *as* $y \in [-5, \infty) \Rightarrow y + 6 \in [1, \infty) \Rightarrow \sqrt{y+6} \geq 1$

$\Rightarrow \frac{-1+\sqrt{y+6}}{3} \geq 0 \Rightarrow x \in R_+$

Also $f(x) = f\left(\frac{-1+\sqrt{y+6}}{3}\right) = 9\left(\frac{-1+\sqrt{y+6}}{3}\right)^2 + 6\left(\frac{-1+\sqrt{y+6}}{3}\right) - 5$

$= 1 - 2\sqrt{y+6} + (y+6) - 2 + 2\sqrt{y+6} - 5 = y$

So for $y \in [-5, \infty)$ there exists $x \in R_+$ such that $(x) = y$.

Hence the function is onto .