DAV PUBLIC SCHOOL,IFFCO,PARADEEP

SUBJECT : MATHEMATICS

TOPIC : MAXIMA AND MINIMA

CLASS-XII

DEPARTMENT OF MATHEMATICS

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MAXIMA AND MINIMA

- In this part of application of derivative , how does the concept of derivative become a tool to determine the maximum and minimum values of a function, turning points of the graph of a function, and the point at which the graph reaches its highest or lowest locally. Further we will also find the absolute maximum and absolute minimum of a function that are necessary for the solution of many applied problems.
- The maximum and the minimum values of a function are together termed as extrema. Pierre De Fermat was one of the Mathematicians to propose a general technique for finding the maxima and minima .

LEARNING OBJECTIVES

- To understand the turning points of a curve .
- To write and explain the definition of absolute maximum and absolute minimum of a function.
- To write the Extreme Value Theorem. Give examples of
  - (i) function(s) that are continuous on an interval (a, b) but do not have an absolute maximum or an absolute minimum;
  - (ii) function(s) that are not continuous on an interval [a, b] and do not have an absolute maximum or absolute minimum.
- To explain a function f continuous on a closed interval [a, b], find the values at which f takes on its absolute maximum and minimum values and find the values of extreme values.
- To write the definition of critical points and explain their importance in finding relative and absolute extreme points.
- To Identify both the candidates for extreme values of a function , if relevant.
- Monotonic Function and first derivative test :
- Given a function f, use the first derivative to identify the intervals on which f is increasing, decreasing.
- Use the first derivative test to determine the nature (relative maximum, relative minimum, neither) of a critical point.
- To learn Secondary Test

A BRIEF INTRODUCTION TO MAXIMA AND MINIMA

- **CLICK HERE**

https://www.youtube.com/watch?v=pvLj1s7SOtk&t=218s

Let us consider the following problems that arise in day today life.

- Suppose as a diligent consumer, you wish to collect data of your cell phone usage (say) for a month.

. You develop a function representing the cell phone usage and then the local maxima/minima give you a fair idea of your cell phone usage which helps you in choosing the most appropriate plan.

- In case we own a company and wish to minimize the cost of production, two types of maxima and minima can prove useful- absolute maxima & minima and local maxima and minima.

- A ball, thrown into the air from a building 60 metres high , travels along a path given by

  $h(x) = 60 + x - \frac{x^2}{60}$, where x is the horizontal distance from the building and $h(x)$ is the height of

  the ball. What is the maximum height the ball will reach?

- An Apache helicopter of enemy is flying along the path given by the curve $f(x) = x^2 + 7$. A soldier, placed at the point $(1,2)$,wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problems, there is something common, $i.e$, we wish to find out the maximum or minimum values of given functions.in order to tackle such problems, we first formally define maximum and minimum values of a function, points of local maxima and minima and test of determining such points.

Definition

Definition 3 Let f be a function defined on an interval I. Then

(a) f is said to have a *maximum value* in I, if there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in$ I.

The number $f(c)$ is called the maximum value of f in I and the point c is called a *point of maximum value* of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in$ I.

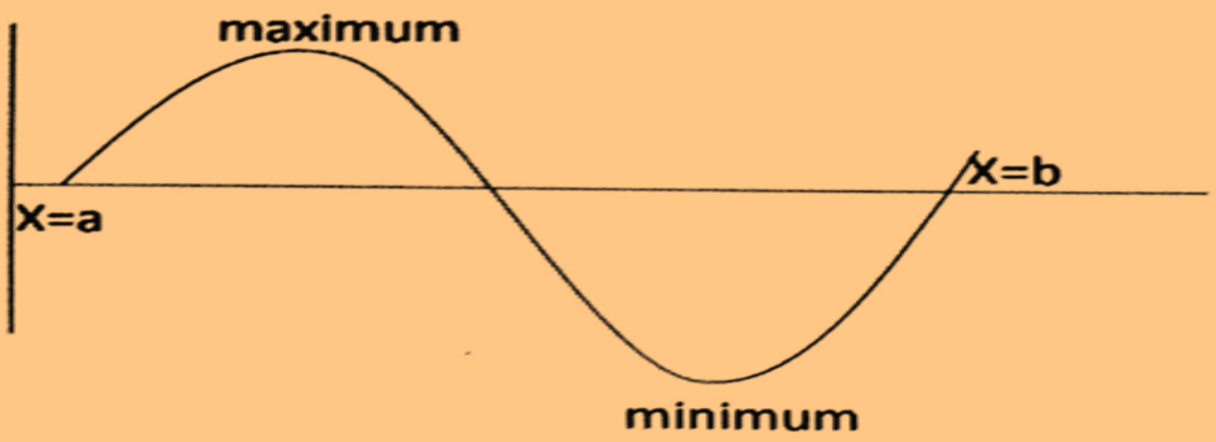
The number $f(c)$, in this case, is called the minimum value of f in I and the point c, in this case, is called a *point of minimum value* of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

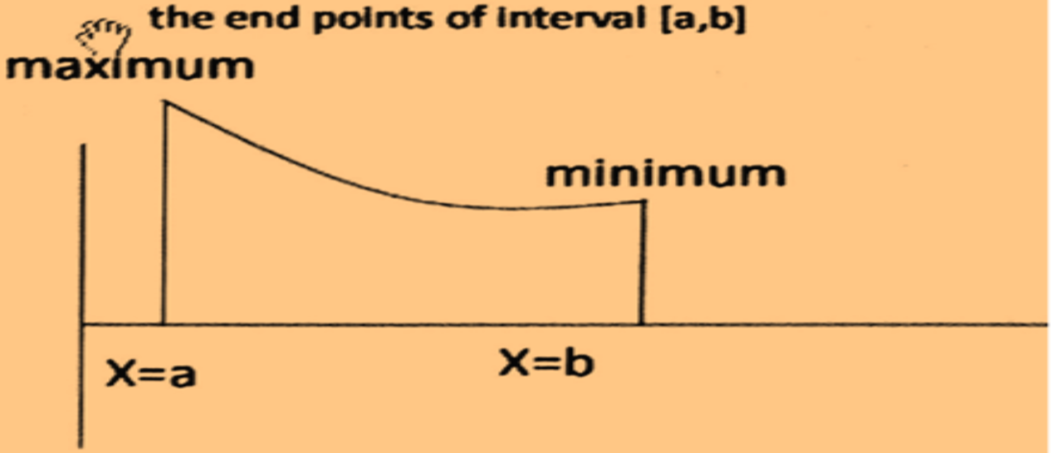
The number $f(c)$, in this case, is called an *extreme value* of f in I and the point is called an *extreme point*.

Observe these graphs

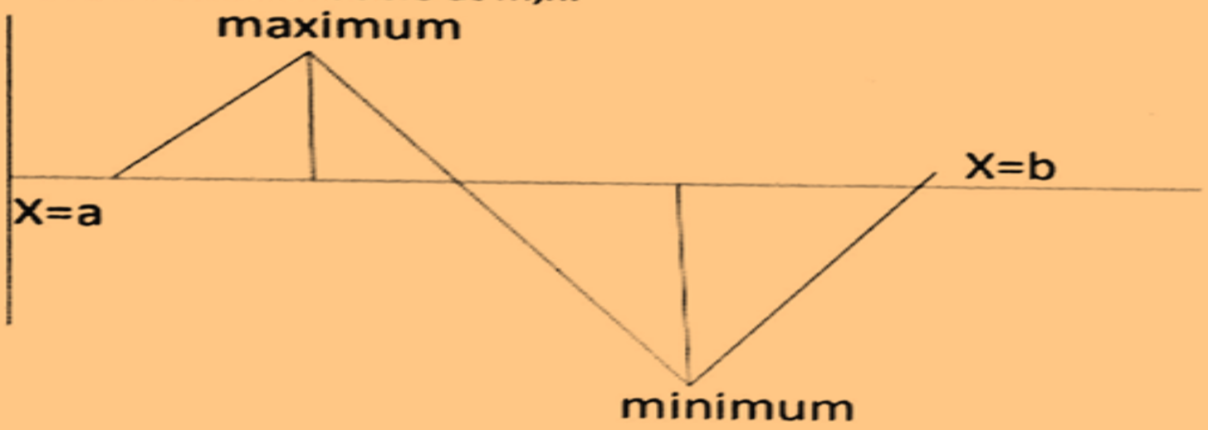
Maximum and minimum are in the interior point of interval [a,b]



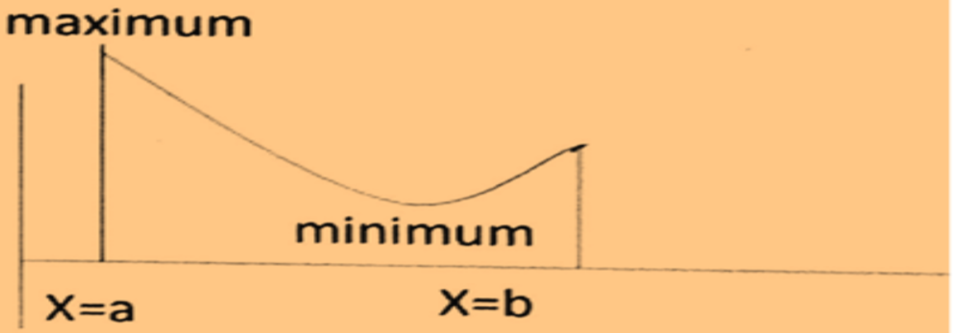
Maximum and minimum are in the end points of interval [a,b]

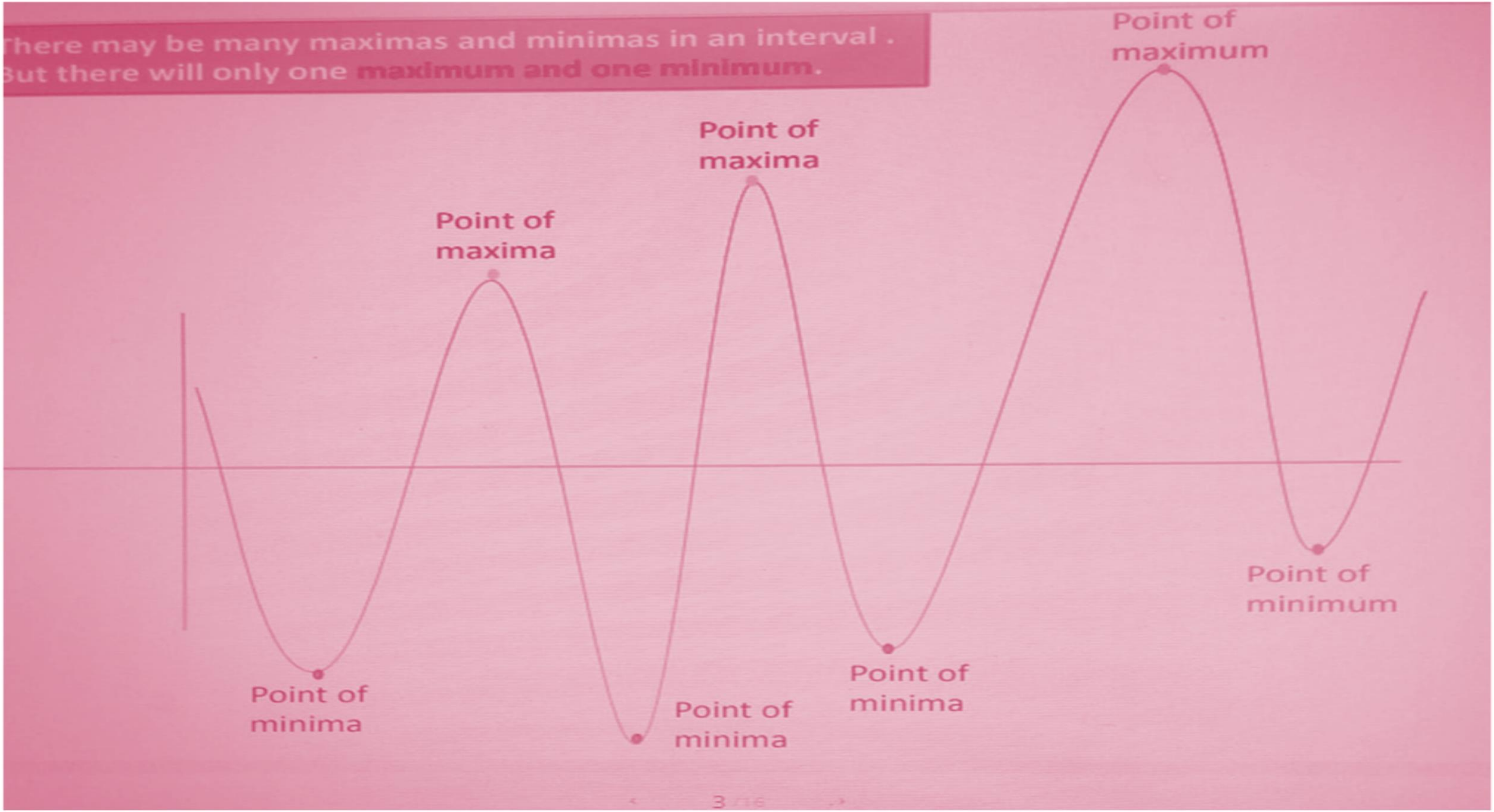


Maximum and minimum are in the interior point of interval [a,b] . At x and y slope is not zero. This function is continuous but not differentiable at m,n.



Maximum at end point, minimum at interior point of interval [a,b]




There may be many maximas and minimas in an interval .
But there will only one maximum and one minimum.
Point of maximum
Point of maxima
Point of maxima
Point of minima
Point of minima
Point of minima
Point of minimum

Find the maximum and minimum values, if any, of the function f given by $f(x) = x^2$, $x \in R$.

Solution From the graph of the given function (Fig 6.10), we have $f(x) = 0$ if $x = 0$. Also

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R}.$$

Therefore, the minimum value of f is 0 and the point of minimum value of f is $x = 0$. Further, it may be observed from the graph of the function that f has no maximum value and hence no point of maximum value of f in $\mathbf{R}$.

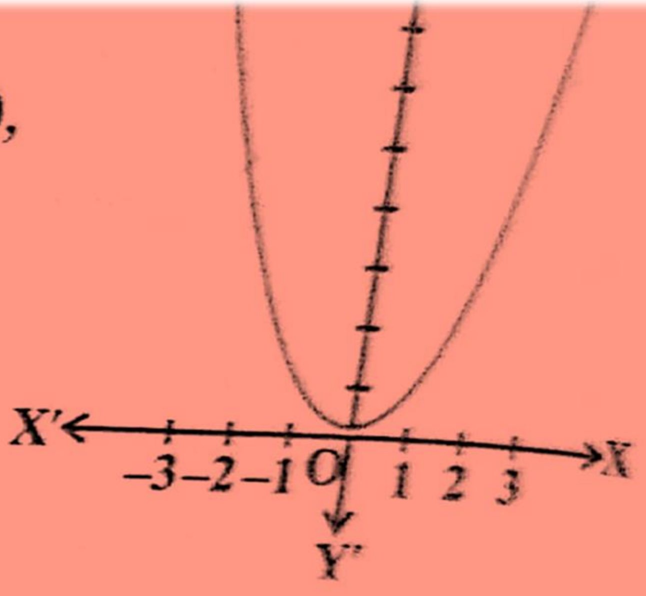


Fig 6.10

Note If we restrict the domain of f to $[-2, 1]$ only, then f will have maximum value $(-2)^2 = 4$ at $x = -2$.

Find the maximum and minimum values, if any, of the function f given by $f(x) = |x|, x \in R$.

Solution From the graph of the given function (Fig 6.11), note that

$f(x) \geq 0$, for all $x \in \mathbf{R}$ and $f(x) = 0$ if $x = 0$.

Therefore, the function f has a minimum value 0 and the point of minimum value of f is $x = 0$. Also, the graph clearly shows that f has no maximum value in $\mathbf{R}$ and hence no point of maximum value in $\mathbf{R}$.

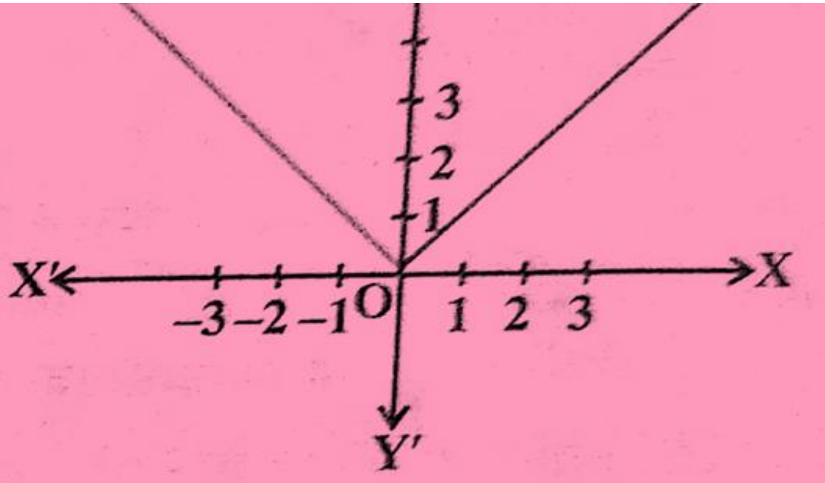


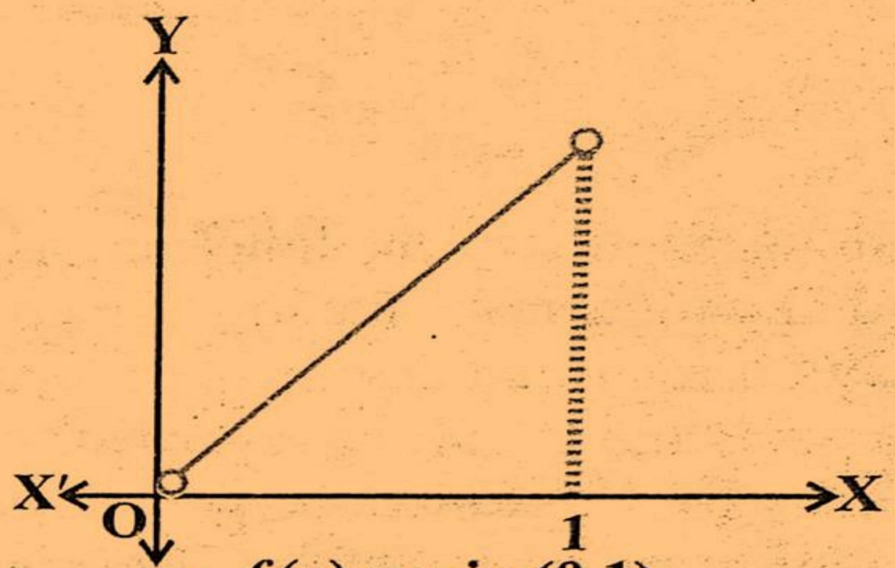
Fig 6.11

Note

(i) If we restrict the domain of f to $[-2, 1]$ only, then f will have maximum value $|-2| = 2$.

Find the maximum and minimum values, if any, of the function f given by $f(x) = x,\ x \in (0,1)$.

Solution The given function is an increasing (strictly) function in the given interval (0, 1). From the graph (Fig 6.12) of the function f, it seems that, it should have the minimum value at a point closest to 0 on its right and the maximum value at a point closest to 1 on its left. Are such points available? Of course, not. It is not possible to locate such points. Infact, if a point x_0 is closest to 0, then we find $\frac{x_0}{2} < x_0$ for all $x_0 \in (0,1)$. Also, if x_1 is closest to 1, then $\frac{x_1+1}{2} > x_1$ for all $x_1 \in (0,1)$.



$f(x) = x$ in (0,1)

Fig 6.12

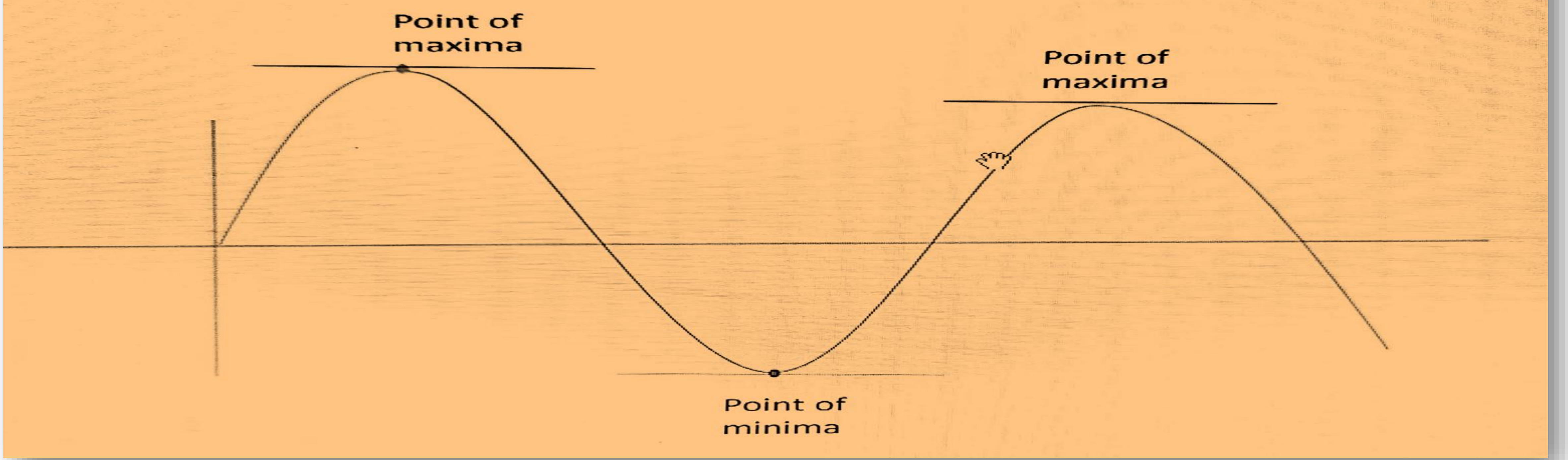
Therefore, the given function has neither the maximum value nor the minimum value in the interval (0,1).

The following points are concluded from the above examples :

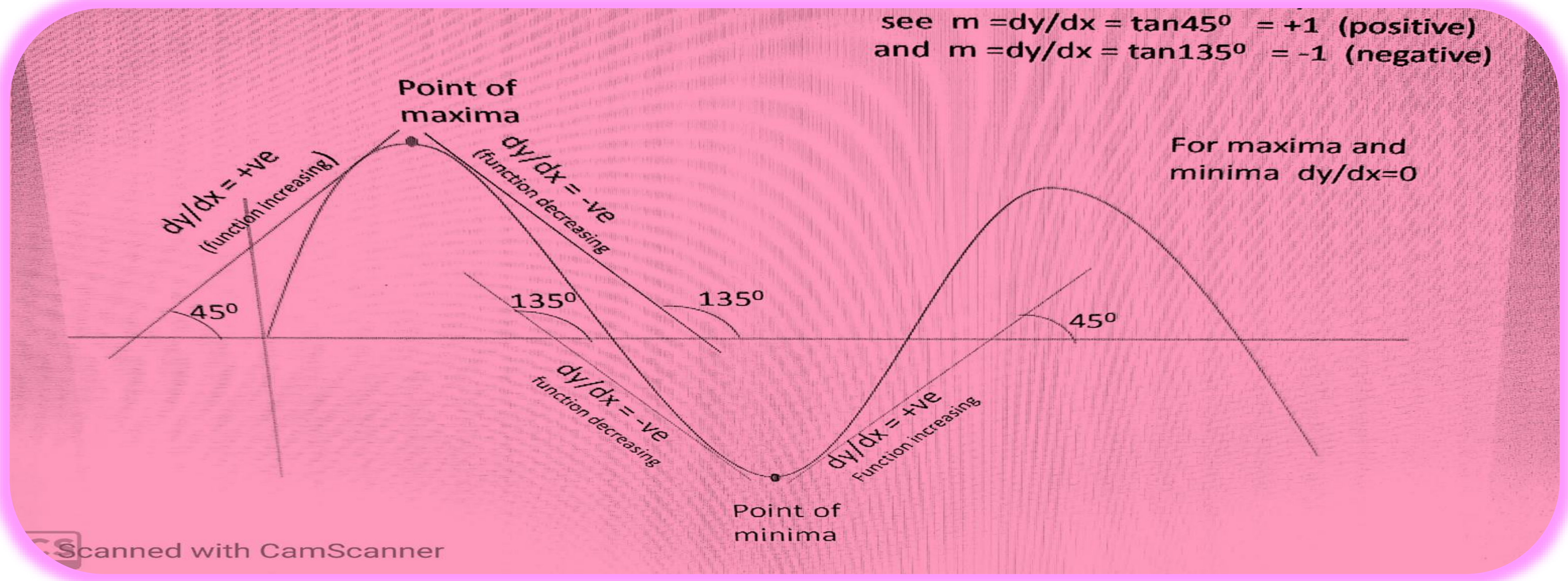
- A function has neither the maximum value nor the minimum value in an open interval.
- Every continuous function on a closed interval has a maximum and a minimum value.
- Every monotonic function assumes its maximum/minimum value at the end points of the domain of definition of the function.
- By a monotonic function f in an interval I, we mean that f is either increasing in I or decreasing in I.

Nature of derivatives at the point of extrema

- For maxima and minima $m = dy/dx = \tan 0^0 = 0$
- $dy/dx = 0$ means tangent is parralel to X –axis.



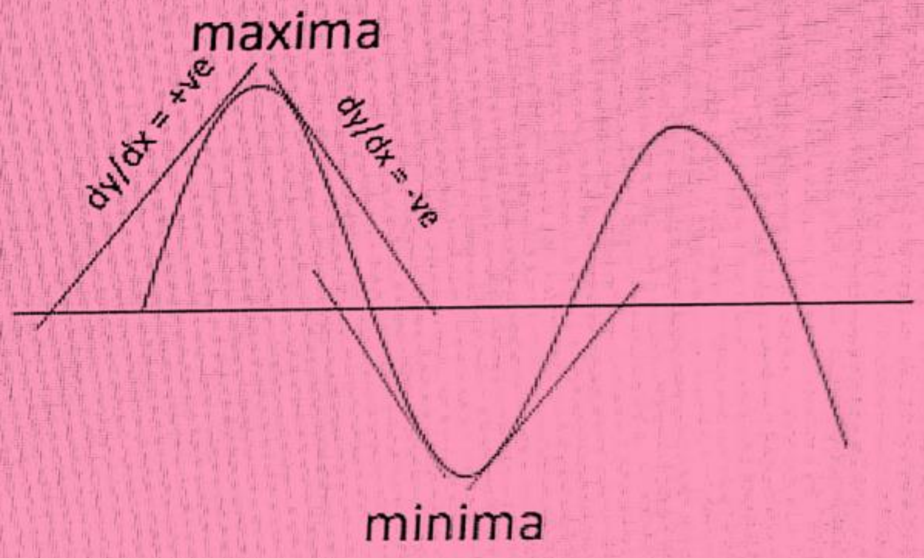
Nature of derivatives at turning points



Nature of derivatives at turning points

- If at left of any point dy/dx = +ve and right of this point dy/dx = -ve then the point will be point of maxima.

- If at left of any point dy/dx = -ve and right of this point dy/dx=+ve then the point will be point of minima.



Definition of local maximum(Relative maximum) and local minimum(Relative minimum)

Definition 4 Let f be a real valued function and let c be an interior point in the domain of f. Then

(a) c is called a point of *local maxima* if there is an $h > 0$ such that

$$f(c) \geq f(x), \text{ for all } x \text{ in } (c-h, c+h)$$

The value $f(c)$ is called the *local maximum value* of f.

(b) c is called a point of *local minima* if there is an $h > 0$ such that

$$f(c) \leq f(x), \text{ for all } x \text{ in } (c-h, c+h)$$

The value $f(c)$ is called the *local minimum value* of f.

Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of f around c will be as shown in Fig 6.14(a). Note that the function f is increasing (i.e., $f'(x) > 0$) in the interval $(c-h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c+h)$.

This suggests that $f'(c)$ must be zero.

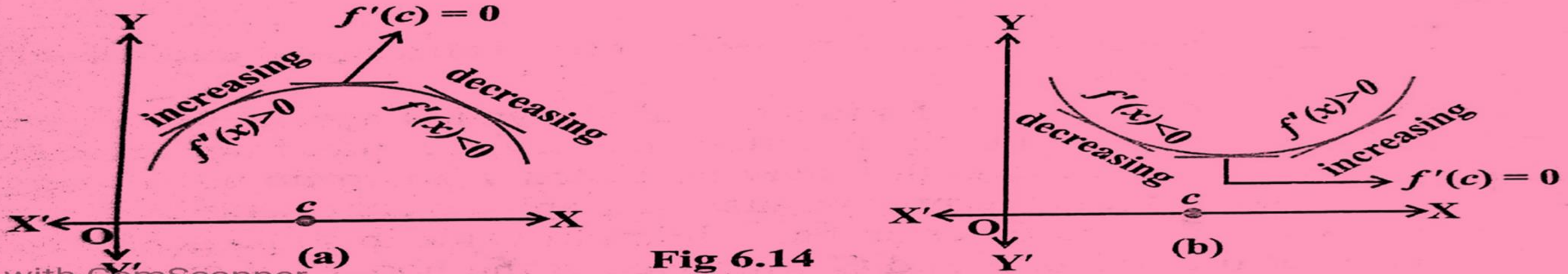


Fig 6.14

Necessary condition of an Extremum

Similarly, if c is a point of local minima of f , then the graph of f around c will be as shown in Fig 6.14(b). Here f is decreasing (i.e., f '(x) < 0) in the interval (c – h, c) and increasing (i.e., f '(x) > 0) in the interval (c, c + h). This again suggest that f '(c) must be zero. The above discussion lead us to the following theorem.

Theorem. If f has an extremum at point $x = c, then\ f'(c) = 0\ or\ f'(c)$ does not exist.

Proof. Let the function has a maximum at $x = c$.

Hence , when $x < c\ then\ f(x) \leq f(c)$

i.e., When $x - c < 0, then\ f(x) - f(c) \leq 0$

i.e., $\frac{f(x)-f(c)}{x-c} \geq 0. hence\ \lim_{n \to c} \frac{f(x)-f(c)}{x-c} \geq 0 \Rightarrow f'(c) \geq 0$ …….(1)

When $x > c\ then\ f(x) \leq f(c)$

$i.e., when\ x-c>0\ then\ f(x)-f(c)\le 0$

Hence, $\frac{f(x)-f(c)}{x-c}\le 0$, $Hence\ \lim_{n\to c}\frac{f(x)-f(c)}{x-c}\le 0\Rightarrow f'(c)\le 0$ …….(2)

From(1)and (2)we see that for maximum at $\mathbf{x=c, f'(c)=0}$. Similarly for minimum value at $\mathbf{x=c, f(c)=0}$.

N.B. if f(x) = |x – 2| then f'(2) does not exist but the function is minimum at x=c.

The converse of the above theorem need not be true, that is, a point at which the derivative vanishes need not be a point of local maxima of local minima. For example, if $f(x)=x^3, then\ f'(x)=3x^2$ and so f'(0)=0.But 0 is neither a point of local maxima nor a point of local minima.

A point at C in the domain of the function f at which either $f'(c) = 0$ or f is not differentiable is called acritical point of f. Note that if f is continous at c and $f'(c) = 0$, then there exists an $h > 0$ such that f is differentiable in the interval $(c - h, c + h)$.

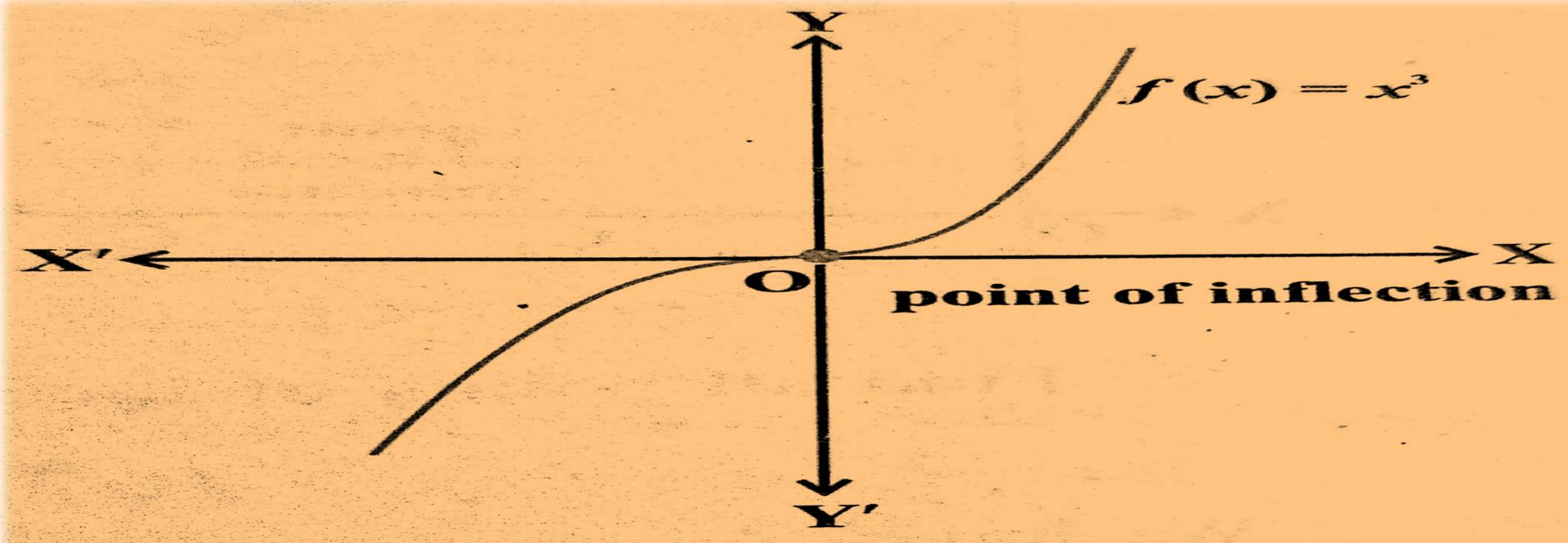


Fig 6.15

Working Rule for finding points of local maxima or points of local minima using only the first order derivative.

Theorem 3 (First Derivative Test) Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflection* (Fig 6.15).

Note: If c is a point of local maxima of f, then $f(c)$ is a local maximum value of f. Similarly, if c is a point of local minima of f, then $f(c)$ is a local minimum value of f.

Figures 6.15 and 6.16, geometrically explain Theorem 3.

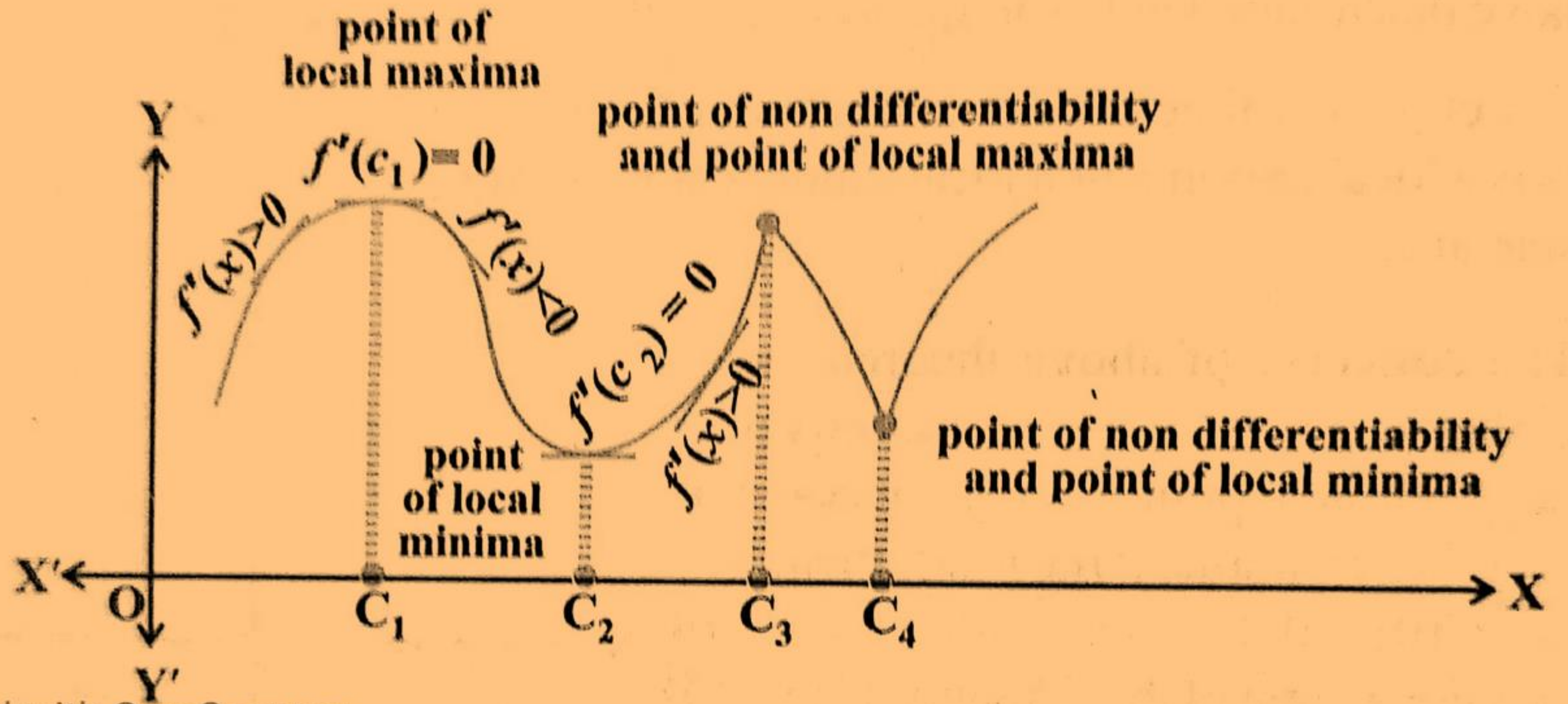


Fig 6.16

Example 29 Find all points of local maxima and local minima of the function f given by

$$f(x) = x^3 - 3x + 3.$$

Solution We have

$$f(x) = x^3 - 3x + 3$$

or $$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$

or $$f'(x) = 0 \text{ at } x = 1 \text{ and } x = -1$$

Thus, $x = \pm 1$ are the only critical points which could possibly be the points of local maxima and/or local minima of f. Let us first examine the point $x = 1$.

Note that for values close to 1 and to the right of 1, $f'(x) > 0$ and for values close to 1 and to the left of 1, $f'(x) < 0$. Therefore, by first derivative test, $x = 1$ is a point of local minima and local minimum value is $f(1) = 1$. In the case of $x = -1$, note that $f'(x) > 0$, for values close to and to the left of -1 and $f'(x) < 0$, for values close to and to the right of -1. Therefore, by first derivative test, $x = -1$ is a point of local maxima and local maximum value is $f(-1) = 5$.

Values of x		Sign of $f'(x) = 3(x-1)(x+1)$
Close to 1	to the right (say 1.1 etc.)	>0
	to the left (say 0.9 etc.)	<0
Close to -1	to the right (say -0.9 etc.)	<0
	to the left (say -1.1 etc.)	>0

Second Derivative Test

Theorem 4 (Second Derivative Test) Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Alternative Method of Second Derivative Test

For knowing point of maxima and minima

a- Find first derivative

b- put dy/dx=0 and find the points for which dy/dx=0

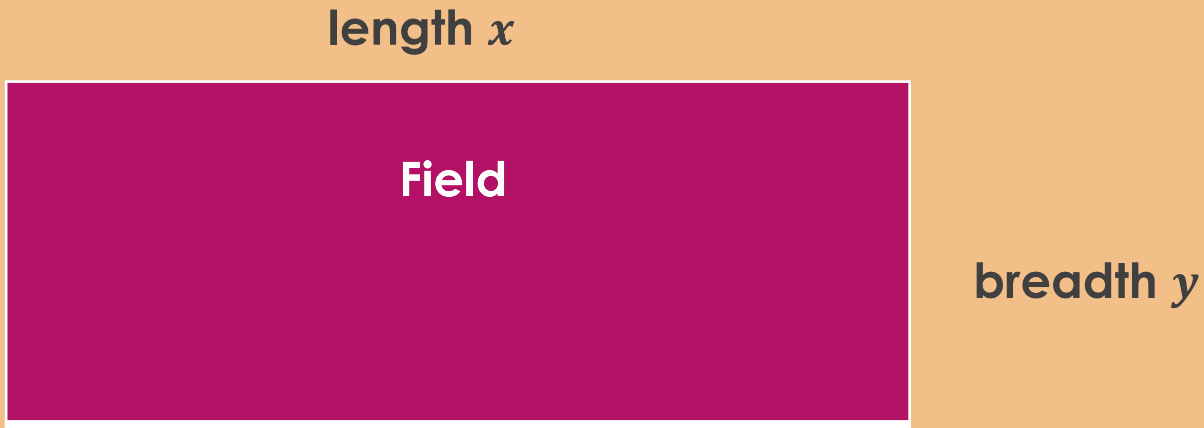
c- now calculate d^2y/dx^2 . If for above any value dy/dx=-ve then it will be the point of maxima. If it is positive then it will point of minima. If d^2y/dx^2 =0 ,then find d^3y/dx^3.

d- If d^3y/dx^3 = +ve or –ve then the point will be neither maxima nor minima. If d^3y/dx^3 = 0, then find d^4y/dx^4. If it is –ve then it will point of maxima and if it is +ve then it will point of minima ,and so on.

Some Examples

- **Find the dimensions of the rectangular field of maximum area which can be fenced by 36 m fence.**

Soln.



$$Let\ length = x, breadth = y.\ Field\ Given\ 2(x+y) = 36.\ So\ x+y = 18$$

Continue........

Area A = $x \, . \, y$,

A = $x(18 - x) \;=\; 18x - x^2$

So dA/dx = d/dx(18x - x^2) = 18- 2x

Put dA/dx = 0 So $18 - 2x \;= 0 \; Or \; x = \; 9$

Now find d^2A/dx^2

d^2A/dx^2 = d/dx(18-2x) = 0-2 = -2 (-ve). So x= 9 will be the point for which area of field will be maximum.

So maximum area = $x(18 - x) \;=\; 9(18 - 9) \;=\; 81m^2$

Continue…

Observe it x+y= 18

Find x.y = maximum Factors for which x+y = 18 may be 1x17 = 17 (product is minimum here)

2x16 = 32

3x15= 45

4x14 = 56

5x13= 65

6x12= 32

7x11= 77

8x10= 80

9x9= 81 (product is maximum here)

Example-2. A cylinder has a fixed surface area. Establish a relation between radius and height of a cylinder for which it's volume is maximum.

Sol. S = $2\pi r$h + 2 πr^2(given)

V = πr^2h (We have to maximise volume.

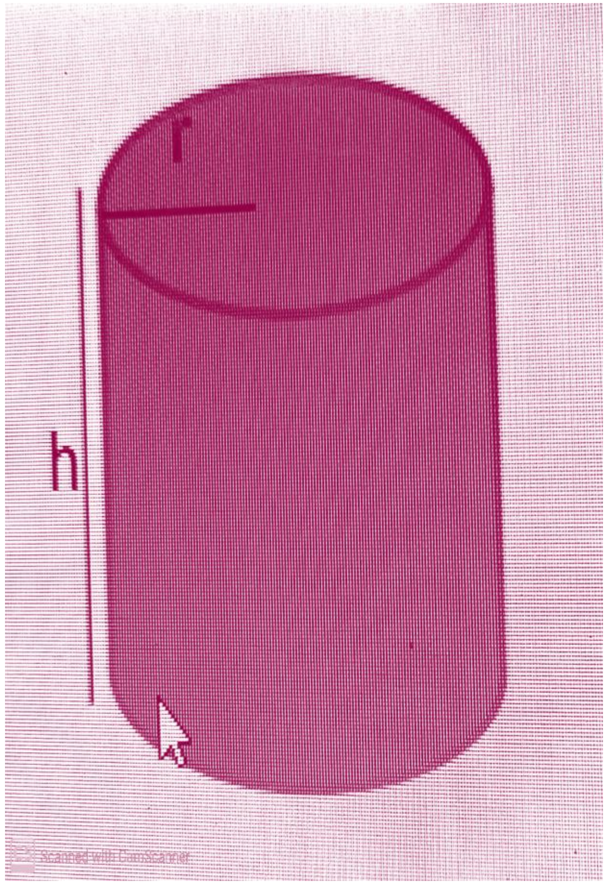
So first reduce variables r and h in either r or h) V= πr^2

(($s - 2\pi r^2$)/2π r)

= r(s - $2\pi r^2$)/2

V= rs - $2\pi r^3$

dv/dr = d/dr(rs - $2\pi r^3$) ⇒ dv/dr = s- $6\pi r^2$



Continue.....

For maxima and minima dv/dr =0 .
So s- $6\pi\ r^2$= 0
or s = $6\pi\ r^2$
Or $2\pi r h + 2\ \pi r^2 = 6\pi\ r^2$
or h= 2r
Now for knowing whether for h=2r volume of cylinder is maximum or minimum,
 we calculate d^2v/dr^2

- $dv/dr = s - 6\pi\ r^2$
- $d^2v/dr^2 = -12\pi\ r$ = -ve
- So volume will maximum at h= 2r

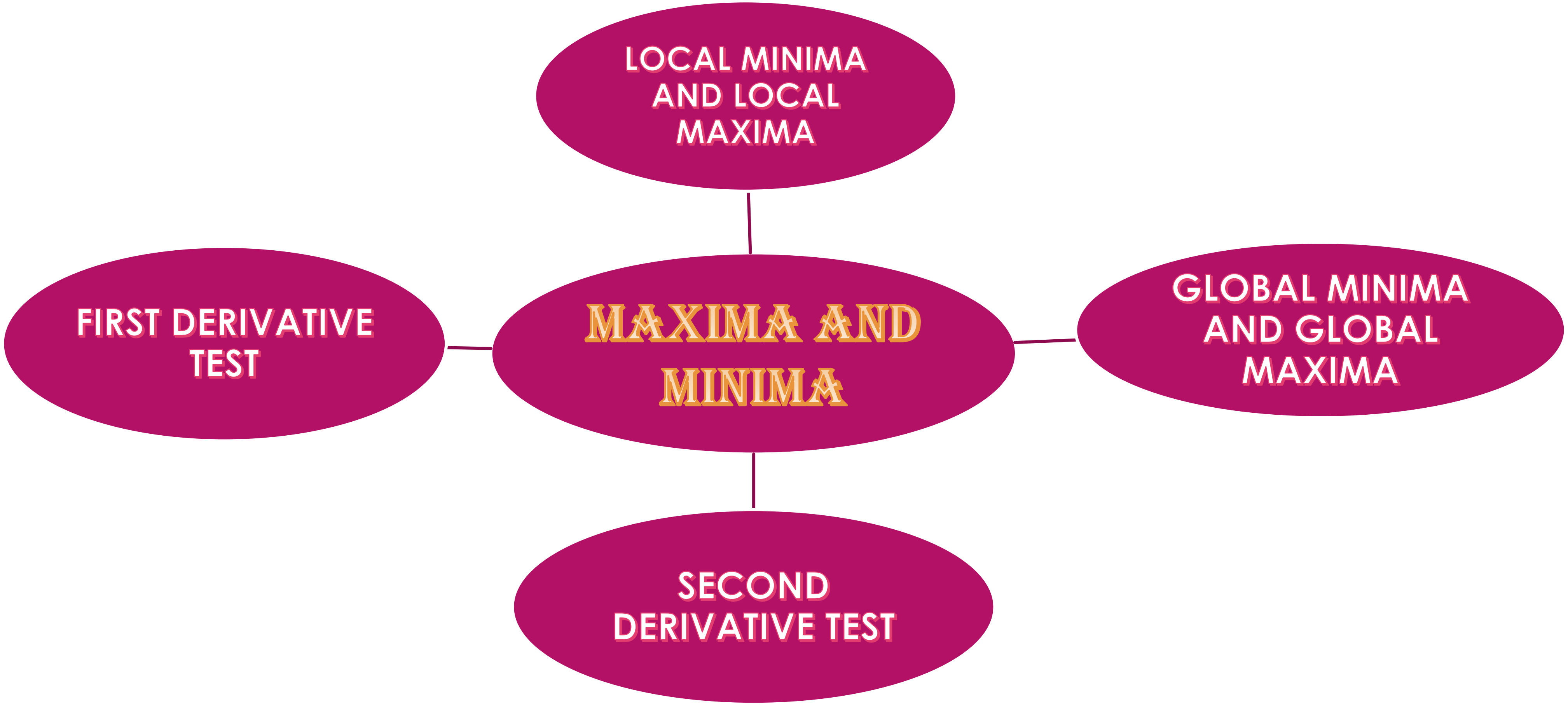
Uses of maxima and minima

- For marketing purposes we require vessels of different shapes for which fabrication cost is less but they could contain more material e.g. 1 litre container of ghee.
- For getting more rectangular land area when total perimeter of land is given.
- In factories using resources so that the fabrication cost of commodity becomes less.

TIPS TO STUDY MAXIMA AND MINIMA

- We can identify the stationary points by looking for the points at which dy/dx = 0.
- All turning points are stationary points but the converse does not hold true, i.e. all stationary points are not turning points.
- A maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.
- A function can have various minimum and maximum values and a minimum value may at times exceed the maximum value.
- The maximum and minimum values of a function always occur alternatively i.e. between every two maximum values, there exists a minimum value and vice-versa.
- If dy/dx = 0 and $d^2y/dx^2 > 0$, then that point must be a point of minima.
- If dy/dx = 0 and $d^2y/dx^2 < 0$, then that point must be a point of maxima.
- Global maximum and minimum in [a, b] would always occur at critical points of f(x) within [a, b] or at the end pints of the interval provided f is continuous in [a, b].
- If f(x) is a continuous function on a closed bounded interval [a,b], then f(x) will have a global maximum and a global minimum on [a,b].
- If the interval is not bounded or closed, then there is no guarantee that a continuous function f(x) will have global extrema.
- If f(x) is differentiable on the interval I, then every global extremum is a local extremum or an endpoint extremum.

CONCEPT MAPPING




Thank You