7.7 Definite Integral

In the previous sections, we have studied about the indefinite integrals and discussed few methods of finding them including integrals of some special functions. In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)\,dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$. Here, we shall consider these two cases separately as discussed below:

7.7.1 *Definite integral as the limit of a sum*

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x = a$ and $x = b$ (Fig 7.2).

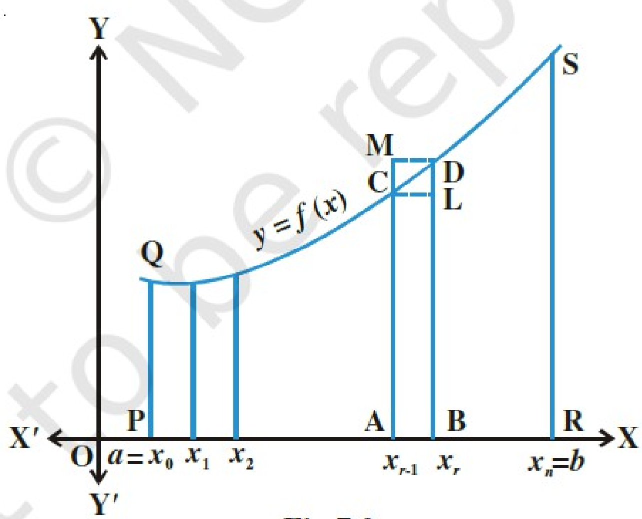

Fig 7.2

Divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1]$, $[x_1, x_2]$,..., $[x_{r-1}, x_r]$, ..., $[x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... , $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \frac{b-a}{h}$. We note that as $n \to \infty$, $h \to 0$.

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots, n$.

From Fig 7.2, we have

area of the rectangle (ABLC) < area of the region (ABDCA) < area of the rectangle (ABDM) ... (1)

Evidently as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h\,[f(x_0) + \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \quad \ldots (2)$$

and

$$S_n = h[f(x_1) + f(x_2) + \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \quad \ldots (3)$$

Here, s_n and S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$, respectively.

In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region PRSQP} < S_n \quad \ldots (4)$$

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region PRSQP} = \int_a^b f(x)dx \quad \ldots (5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h]$$

or

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h] \quad \ldots (6)$$

where

$$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression (6) is known as the definition of definite integral as the *limit of sum.*

Remark The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)\,dt$ or $\int_a^b f(u)\,du$ instead of $\int_a^b f(x)\,dx$. Hence, the variable of integration is called a *dummy variable*.

Example 25 Find $\int_0^2 (x^2+1)\,dx$ as the limit of a sum.

Solution By definition

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+...+f(a+(n-1)\,h],$$

where, $$h=\frac{b-a}{n}$$

In this example, $a = 0$, $b = 2$, $f(x) = x^2 + 1$, $h=\frac{2-0}{n}=\frac{2}{n}$

Therefore,

$$\int_0^2 (x^2+1)\,dx = 2\lim_{n\to\infty}\frac{1}{n}[f(0)+f(\frac{2}{n})+f(\frac{4}{n})+...+f(\frac{2\,(n-1)}{n})]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[1+(\frac{2^2}{n^2}+1)+(\frac{4^2}{n^2}+1)+...+\left(\frac{(2n-2)^2}{n^2}+1\right)]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[\underbrace{(1+1+...+1)}_{n\text{-terms}}+\frac{1}{n^2}(2^2+4^2+...+(2n-2)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2^2}{n^2}(1^2+2^2+...+(n-1)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{4}{n^2}\frac{(n-1)\,n\,(2n-1)}{6}]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2}{3}\frac{(n-1)\;(2n-1)}{n}]$$

$$= 2\lim_{n\to\infty}[1+\frac{2}{3}(1-\frac{1}{n})\;(2-\frac{1}{n})] = 2\;[1+\frac{4}{3}] = \frac{14}{3}$$

Example 26 Evaluate $\int_0^2 e^x \, dx$ as the limit of a sum.

Solution By definition

$$\int_0^2 e^x \, dx = (2-0) \lim_{n\to\infty} \frac{1}{n}\left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}}\right]$$

Using the sum to n terms of a G.P., where $a = 1$, $r = e^{\frac{2}{n}}$, we have

$$\int_0^2 e^x \, dx = 2 \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right] = 2 \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{2\,(e^2-1)}{\lim\limits_{n\to\infty}\left[\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right]\cdot 2} = e^2 - 1 \qquad \left[\text{using } \lim_{h\to 0}\frac{(e^h-1)}{h} = 1\right]$$

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x \, dx$ **2.** $\int_0^5 (x+1) \, dx$ **3.** $\int_2^3 x^2 \, dx$

4. $\int_1^4 (x^2 - x) \, dx$ **5.** $\int_{-1}^1 e^x \, dx$ **6.** $\int_0^4 (x + e^{2x}) \, dx$

7.8 Fundamental Theorem of Calculus

7.8.1 *Area function*

We have defined $\int_a^b f(x)\, dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and x-axis. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\, dx$ represents the area of the light shaded region

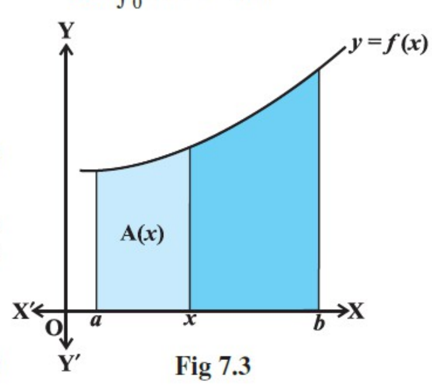

Fig 7.3

in Fig 7.3 [Here it is assumed that $f(x) > 0$ for $x \in [a, b]$, the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of x.

In other words, the area of this shaded region is a function of x. We denote this function of x by $A(x)$. We call the function $A(x)$ as *Area function* and is given by

$$\mathbf{A}(x) = \int_a^x f(x)\,dx \qquad \ldots (1)$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

7.8.2 *First fundamental theorem of integral calculus*

Theorem 1 Let f be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $\mathbf{A'(x) = f(x)}$, for all $\mathbf{x \in [a, b]}$.

7.8.3 *Second fundamental theorem of integral calculus*

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 2 Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

Remarks

(i) In words, the Theorem 2 tells us that $\int_a^b f(x)\,dx$ = (value of the anti derivative F of f at the upper limit b – value of the same anti derivative at the lower limit a).

(ii) This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

(iii) The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.

(iv) In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in $[a, b]$. For instance, the consideration of definite integral $\int_{-2}^{3} x(x^2-1)^{\frac{1}{2}}\,dx$ is erroneous since the function f expressed by $f(x) = x(x^2-1)^{\frac{1}{2}}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

Steps for calculating $\int_a^b f(x)\,dx$.

(i) Find the indefinite integral $\int f(x)\,dx$. Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get

$\int_a^b f(x)\,dx = [\mathrm{F}(x)+\mathrm{C}]_a^b = [\mathrm{F}(b)+\mathrm{C}] - [\mathrm{F}(a)+\mathrm{C}] = \mathrm{F}(b) - \mathrm{F}(a)$.

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate F(b) – F(a) = $[\mathrm{F}(x)]_a^b$, which is the value of $\int_a^b f(x)\,dx$.

We now consider some examples

Example 27 Evaluate the following integrals:

(i) $\int_2^3 x^2\,dx$ (ii) $\int_4^9 \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx$

(iii) $\int_1^2 \frac{x\,dx}{(x+1)(x+2)}$ (iv) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\,dt$

Solution

(i) Let $\mathrm{I} = \int_2^3 x^2\,dx$. Since $\int x^2\,dx = \frac{x^3}{3} = \mathrm{F}(x)$,

Therefore, by the second fundamental theorem, we get

$$\mathrm{I} = \mathrm{F}(3) - \mathrm{F}(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

(ii) Let $\mathrm{I} = \int_4^9 \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx$. We first find the anti derivative of the integrand.

Put $30 - x^{\frac{3}{2}} = t$. Then $-\frac{3}{2}\sqrt{x}\,dx = dt$ or $\sqrt{x}\,dx = -\frac{2}{3}dt$

Thus, $\int \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx = -\frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right] = \frac{2}{3}\left[\frac{1}{(30-x^{\frac{3}{2}})}\right] = \mathrm{F}(x)$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(9) - F(4) = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30-27)} - \frac{1}{30-8}\right] = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right] = \frac{19}{99}$$

(iii) Let $I = \int_1^2 \frac{x\,dx}{(x+1)(x+2)}$

Using partial fraction, we get $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

So $$\int \frac{x\,dx}{(x+1)(x+2)} = -\log|x+1| + 2\log|x+2| = F(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(2) - F(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3]$$

$$= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right)$$

(iv) Let $I = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\, dt$. Consider $\int \sin^3 2t \cos 2t\, dt$

Put $\sin 2t = u$ so that $2\cos 2t\, dt = du$ or $\cos 2t\, dt = \frac{1}{2}\, du$

So $$\int \sin^3 2t \cos 2t\, dt = \frac{1}{2}\int u^3 du$$

$$= \frac{1}{8}[u^4] = \frac{1}{8}\sin^4 2t = F(t) \text{ say}$$

Therefore, by the second fundamental theorem of integral calculus

$$I = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{8}\left[\sin^4\frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

1. $\int_{-1}^{1}(x+1)\,dx$ **2.** $\int_{2}^{3}\frac{1}{x}\,dx$ **3.** $\int_{1}^{2}(4x^3-5x^2+6x+9)\,dx$

4. $\int_{0}^{\frac{\pi}{4}}\sin 2x\,dx$ **5.** $\int_{0}^{\frac{\pi}{2}}\cos 2x\,dx$ **6.** $\int_{4}^{5}e^{x}dx$ **7.** $\int_{0}^{\frac{\pi}{4}}\tan x\,dx$

8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\operatorname{cosec} x\,dx$ **9.** $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$ **10.** $\int_{0}^{1}\frac{dx}{1+x^2}$ **11.** $\int_{2}^{3}\frac{dx}{x^2-1}$

12. $\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx$ **13.** $\int_{2}^{3}\frac{x\,dx}{x^2+1}$ **14.** $\int_{0}^{1}\frac{2x+3}{5x^2+1}\,dx$ **15.** $\int_{0}^{1}x\,e^{x^2}\,dx$

16. $\int_{1}^{2}\frac{5x^2}{x^2+4x+3}$ **17.** $\int_{0}^{\frac{\pi}{4}}(2\sec^2 x+x^3+2)\,dx$ **18.** $\int_{0}^{\pi}(\sin^2\frac{x}{2}-\cos^2\frac{x}{2})\,dx$

19. $\int_{0}^{2}\frac{6x+3}{x^2+4}\,dx$ **20.** $\int_{0}^{1}(x\,e^{x}+\sin\frac{\pi x}{4})\,dx$

Choose the correct answer in Exercises 21 and 22.

21. $\int_{1}^{\sqrt{3}}\frac{dx}{1+x^2}$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

22. $\int_{0}^{\frac{2}{3}}\frac{dx}{4+9x^2}$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

7.9 Evaluation of Definite Integrals by Substitution

In the previous sections, we have discussed several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution.

To evaluate $\int_a^b f(x)\,dx$, by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute, $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form.
2. Integrate the new integrand with respect to the new variable without mentioning the constant of integration.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

Note In order to quicken this method, we can proceed as follows: After performing steps 1, and 2, there is no need of step 3. Here, the integral will be kept in the new variable itself, and the limits of the integral will accordingly be changed, so that we can perform the last step.

Let us illustrate this by examples.

Example 28 Evaluate $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$.

Solution Put $t = x^5 + 1$, then $dt = 5x^4\,dx$.

Therefore, $$\int 5x^4\sqrt{x^5+1}\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$$

Hence, $$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{2}{3}\left[(1^5+1)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Alternatively, first we transform the integral and then evaluate the transformed integral with new limits.

Let $t = x^5 + 1$. Then $dt = 5\,x^4\,dx$.

Note that, when $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

Thus, as x varies from – 1 to 1, t varies from 0 to 2

Therefore $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \int_0^2 \sqrt{t}\,dt$

$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Example 29 Evaluate $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$

Solution Let $t = \tan^{-1}x$, then $dt = \frac{1}{1+x^2}\,dx$. The new limits are, when $x = 0$, $t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$. Thus, as x varies from 0 to 1, t varies from 0 to $\frac{\pi}{4}$.

Therefore $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} t\,dt \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

1. $\int_0^1 \frac{x}{x^2+1}\,dx$ **2.** $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$ **3.** $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

4. $\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$) **5.** $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

6. $\int_0^2 \frac{dx}{x+4-x^2}$ **7.** $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$ **8.** $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right)e^{2x}\,dx$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\,dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

10. If $f(x) = \int_0^x t\sin t\,dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$ (B) $x\sin x$

(C) $x\cos x$ (D) $\sin x + x\cos x$

7.10 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

P_0 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$

P_2 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

P_3 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P_4 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(Note that P_4 is a particular case of P_3)

P_5 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P_6 : $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$ and

0 if $f(2a-x) = -f(x)$

P_7 : (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function, i.e., if $f(-x) = f(x)$.

(ii) $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.

We give the proofs of these properties one by one.

Proof of P_0 It follows directly by making the substitution $x = t$.

Proof of P_1 Let F be anti derivative of f. Then, by the second fundamental theorem of calculus, we have $\int_a^b f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\,dx$

Here, we observe that, if $a = b$, then $\int_a^a f(x)\,dx = 0$.

Proof of P_2 Let F be anti derivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{... (1)}$$

$$\int_a^c f(x)\,dx = F(c) - F(a) \quad \text{... (2)}$$

and

$$\int_c^b f(x)\,dx = F(b) - F(c) \quad \text{... (3)}$$

Adding (2) and (3), we get $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(b) - F(a) = \int_a^b f(x)\,dx$

This proves the property P_2.

Proof of P_3 Let $t = a + b - x$. Then $dt = -\,dx$. When $x = a$, $t = b$ and when $x = b$, $t = a$. Therefore

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \quad \text{(by } P_1\text{)}$$

$$= \int_a^b f(a+b-x)\,dx \text{ by } P_0$$

Proof of P_4 Put $t = a - x$. Then $dt = -\,dx$. When $x = 0$, $t = a$ and when $x = a$, $t = 0$. Now proceed as in P_3.

Proof of P_5 Using P_2, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$.

Let $t = 2a - x$ in the second integral on the right hand side. Then $dt = -\,dx$. When $x = a$, $t = a$ and when $x = 2a$, $t = 0$. Also $x = 2a - t$.

Therefore, the second integral becomes

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

Hence $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Proof of P_6 Using P_5, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ ... (1)

Now, if $f(2a - x) = f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx,$$

and if $f(2a - x) = -f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

Proof of P_7 Using P_2, we have

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \text{. Then}$$

Let $t = -x$ in the first integral on the right hand side. $dt = -\,dx$. When $x = -a$, $t = a$ and when $x = 0$, $t = 0$. Also $x = -t$.

Therefore
$$\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx$$
$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{(by } P_0\text{)} \quad \ldots (1)$$

(i) Now, if f is an even function, then $f(-x) = f(x)$ and so (1) becomes
$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(ii) If f is an odd function, then $f(-x) = -f(x)$ and so (1) becomes
$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Example 30 Evaluate $\int_{-1}^{2} \left| x^3 - x \right| dx$

Solution We note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and that $x^3 - x \geq 0$ on $[1, 2]$. So by P_2 we write
$$\int_{-1}^{2} \left| x^3 - x \right| dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$
$$= \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$
$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$
$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$
$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

Example 31 Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx$

Solution We observe that $\sin^2 x$ is an even function. Therefore, by P_7 (i), we get
$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{(1-\cos 2x)}{2}\,dx = \int_0^{\frac{\pi}{4}} (1-\cos 2x)\,dx$$

$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{\pi}{4} - \frac{1}{2}$$

Example 32 Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

Solution Let $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$. Then, by P_4, we have

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\,dx}{1+\cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1+\cos^2 x} = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - I$$

or $$2I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

or $$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$. Therefore, (by P_1) we get

$$I = \frac{-\pi}{2}\int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi\int_0^{1} \frac{dt}{1+t^2} \text{ (by } P_7\text{, since } \frac{1}{1+t^2} \text{ is even function)}$$

$$= \pi\left[\tan^{-1} t\right]_0^1 = \pi\left[\tan^{-1} 1 - \tan^{-1} 0\right] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$

Example 33 Evaluate $\int_{-1}^{1} \sin^5 x\cos^4 x\,dx$

Solution Let $I = \int_{-1}^{1} \sin^5 x\cos^4 x\,dx$. Let $f(x) = \sin^5 x\cos^4 x$. Then

$f(-x) = \sin^5(-x)\cos^4(-x) = -\sin^5 x\cos^4 x = -f(x)$, i.e., f is an odd function. Therefore, by P_7 (ii), $I = 0$

Example 34 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

Solution Let $\text{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ ... (1)

Then, by P_4

$$\text{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^4 (\frac{\pi}{2} - x)}{\sin^4 (\frac{\pi}{2} - x) + \cos^4 (\frac{\pi}{2} - x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x} dx \quad ... (2)$$

Adding (1) and (2), we get

$$2\text{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Hence $\text{I} = \frac{\pi}{4}$

Example 35 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}}$

Solution Let $\text{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}}$ ... (1)

Then, by P_3

$$\text{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} dx}{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad ... (2)$$

Adding (1) and (2), we get

$$2\text{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \text{ Hence } \text{I} = \frac{\pi}{12}$$

Example 36 Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Solution Let $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Then, by P_4

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

Adding the two values of I, we get

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x \cos x + \log 2 - \log 2) \, dx \text{ (by adding and subtracting } \log 2)$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx \quad \text{(Why?)}$$

Put $2x = t$ in the first integral. Then $2\,dx = dt$, when $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$.

Therefore $$2I = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2} \log 2$$

$$= \frac{2}{2} \int_0^{\frac{\pi}{2}} \log \sin t \, dt - \frac{\pi}{2} \log 2 \quad [\text{by } P_6 \text{ as } \sin(\pi - t) = \sin t)$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2} \log 2 \quad \text{(by changing variable } t \text{ to } x)$$

$$= I - \frac{\pi}{2} \log 2$$

Hence $$\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{-\pi}{2} \log 2.$$

EXERCISE 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1. $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$

3. $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x\, dx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x\, dx}{\sin^5 x+\cos^5 x}$

5. $\int_{-5}^{5} |x+2|\, dx$

6. $\int_2^8 |x-5|\, dx$

7. $\int_0^1 x(1-x)^n\, dx$

8. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

9. $\int_0^2 x\sqrt{2-x}\, dx$

10. $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\, dx$

11. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

12. $\int_0^{\pi} \frac{x\, dx}{1+\sin x}$

13. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx$

14. $\int_0^{2\pi} \cos^5 x\, dx$

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}\, dx$

16. $\int_0^{\pi} \log(1+\cos x)\, dx$

17. $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx$

18. $\int_0^4 |x-1|\, dx$

19. Show that $\int_0^a f(x)g(x)\, dx = 2\int_0^a f(x)\, dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Choose the correct answer in Exercises 20 and 21.

20. The value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$ is

(A) 0 (B) 2 (C) π (D) 1

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(A) 2 (B) $\frac{3}{4}$ (C) 0 (D) –2